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SECOND QUANTIZATION OF THE WILSON LOOP

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Abstract

Treating the QCD Wilson loop as amplitude for the propagation of the first quantized particle we develop the second quantization of the same propagation. The operator of the particle position $\hat{\mathcal{X}}_\mu$ (the endpoint of the "open string") is introduced as a limit of the large N Hermitean matrix. We then derive the set of equations for the expectation values of the vertex operators $\langle V(k_1)\dots V(k_n)\rangle$. The remarkable property of these equations is that they can be expanded at small momenta (less than the QCD mass scale), and solved for expansion coefficients. This provides the relations for multiple commutators of position operator, which can be used to construct this operator. We employ the noncommutative probability theory and find the expansion of the operator $\hat{\mathcal{X}}_\mu$ in terms of products of creation operators $a^\dagger_\mu$. In general, there are some free parameters left in this expansion. In two dimensions we fix parameters uniquely from the symplectic invariance. The Fock space of our theory is much smaller than that of perturbative QCD, where the creation and annihilation operators were labelled by continuous momenta. In our case this is a space generated by $d = 4$ creation operators. The corresponding states are given by all sentences made of the four letter words. We discuss the implication of this construction for the mass spectra of mesons and glueballs.

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1 Introduction

The interpretation of the Wilson loop $W(C)$ as an amplitude of the charge propagation has a long history. Already in early Feynman's work the abelian phase factor was interpreted this way. Wilson elaborated on that in his famous paper "Confinement of Quarks", taking the first quantized picture.

The quark propagator in this picture is given by the sum over closed loops C of the loop amplitudes $\sum_C W(C)G(C)$, where $G(C)$ is the amplitude for the free quark. In the large N limit there are no dynamical quarks, which makes this picture especially attractive. There is a tempting analogy with the string theory. In fact, the string theory originated as the planar diagram theory of quark confinement and was boosted by the 't Hooft's discovery [1] of the large N limit.

Now, what would be the second quantized picture of this process? This is not just the matter of mathematical curiosity. The development of Feynman-Wilson picture for the last 20 years led us to the dead end. We derived[2] nonlinear loop equation for $W(C)$, but this equation turned out to be too hard to solve.

In two dimensions, where there are only global degrees of freedom, related to topology of the loop, the loop equations were solved[4]. In four dimensions the best bet was the Eguchi-Kawai reduction, which, unfortunately worked only on the lattice. Taking the continuum limit in the EK model appears to be as hard as in the original partition function of the lattice theory. This forces us to look for the new approaches, such as the second quantized picture.

1.1 Classical vs Quantum Loops

The second quantization of the usual field theory involves the following steps

$$fields \Rightarrow operators \Rightarrow commutators \Rightarrow H\Psi = E\Psi. \tag{1}$$

We would like to follow these steps in QCD for the particle position field $\hat{X}$. The parametric invariance of the Wilson loop would make this one dimensional topological field theory. The Hamiltonian H vanishes in such theory, as the field operators do not depend of the (proper) time.

To understand this striking phenomenon, let us consider the following simple example: Wilson loop in constant abelian electromagnetic field $F_{\mu\nu}$

$$W(C) = \exp\left(\frac{1}{2}\imath\, F_{\mu\nu} \oint_C x_\mu\, dx_\nu\right). \tag{2}$$

The exponential here represents symplectic form which immediately leads to the commutation relations

$$[\hat{X}_\mu, \hat{X}_\nu] = \imath\ (F^{-1})_{\mu\nu}. \tag{3}$$

of $\frac{1}{2}d$ harmonic oscillators.

This corresponds to the following momentum space path integral[1]

$$W(C) \propto \frac{1}{N}\int Dp(.)\delta^d(p(s_0))\ \exp\left(\imath \oint dx_\mu p_\mu\right) \mathrm{tr}\ \hat{T}\exp\left(\imath \oint d\,p_\mu\, \hat{X}_\mu\right), \tag{4}$$

One may regard $p'_\mu(s)$ as Lagrange multiplier for the constraint on the coordinate $x_\mu(s) \approx \hat{X}_\mu$ modulo total translations $\delta x_\mu(s) = const$. The constraint

$$p(s_0) = 0 \tag{5}$$

is needed to eliminate the translations of $p(s)$. The choice of s_0 is arbitrary, it is a matter of convenience.

The trace is calculable for the oscillator, it yields

$$\mathrm{tr}\ \hat{T}\exp\left(\imath \oint d\,p_\mu\, \hat{X}_\mu\right) \propto \exp\left(-\frac{1}{2}\imath\,(F^{-1})_{\mu\nu} \oint p_\mu dp_\nu\right). \tag{6}$$

The integration over $p(.)$ correctly reproduces the initial formula for $W(C)$. For the readers reference, this functional integral is computed in Appendix A, including the normalization (irrelevant for our present purposes).

Note, that the operators $\hat{X}_\mu$ do not commute, and, hence, cannot be simultaneously diagonalized. So, we cannot integrate out the Lagrange multiplier and claim that $x_\mu(s)$ coincides with one of the eigenvalues of $\hat{X}_\mu$.

[1] Later we shall slightly modify this formula, to take into account the discontinuities of the local momentum $p(s)$.

The Wilson loop enters the physical observables inside the path integral, such as

$$\int_0^\infty dT \int Dx(.)\delta^d(x(s_0)) \exp\left(-\int_0^T dt \left(\frac{1}{4}\dot{x}^2 + m^2\right)\right) W(C) \tag{7}$$

for the scalar particle propagator. With our representation after integrating out $x(.)$ this becomes

$$\int_0^\infty dT \int Dp(.)\,\hat{T} \exp\left(\int_0^T dt \left(\imath\, \dot{p}_\mu \hat{X}_\mu - H(p)\right)\right) \tag{8}$$

with the (proper time) Hamiltonian

$$H = m^2 + p_\mu^2. \tag{9}$$

For the Dirac particle we get the same formula with the Dirac Hamiltonian

$$H = m + \imath\, \gamma_\mu p_\mu \tag{10}$$

The only difference with the usual Hamiltonian dynamics is the replacement of the particle position $x(t)$ by the operator$\hat{X}$ in the $\dot{p}x$ term. The implications of this replacements will be studied later.

1.2 Master Field and Eguchi-Kawai Reduction

In a way, our approach is an implementation of the Witten's master field idea[5]. He suggested to look for the classical x-independent field $\hat{A}_\mu$ in coordinate representation

$$W(C) = \frac{1}{N} \mathrm{tr}\, \hat{T} \exp\left(\oint_C dx_\mu \hat{A}_\mu\right). \tag{11}$$

Clearly, the naive implementation, with $\hat{A}_\mu$ satisfying the Yang-Mills equations, does not work. There are corrections from the fluctuations of the gauge field. These corrections are taken into account in the loop equations[2, 3], so it is worth trying to find effective master field equation for $\hat{A}_\mu$ from the loop equations.

In some broad sense the master field always exists. It merely represents the covariant derivative operator

$$\nabla_\mu = -\imath\, \mathcal{P}_\mu + \mathcal{A}_\mu(0), \tag{12}$$

where $\mathcal{P}_\mu$ is momentum operator and $\mathcal{A}_\mu$ is the gauge field operator in the Hilbert space. As one may readily check, the matrix trace of the ordered exponential of this operator reduces to the operator version of the loop exponential times the unit operator in Hilbert space

$$\mathrm{tr}\, \hat{T} \exp\left(\int_0^0 \nabla_\mu dx_\mu\right) = \mathrm{tr}\, \hat{T} \exp\left(\int_0^0 \mathcal{A}_\mu(x) dx_\mu\right), \tag{13}$$
$$\mathcal{A}_\mu(x) = \exp\left(-\imath\, \mathcal{P}_\nu x_\nu\right) \mathcal{A}_\mu(0) \exp\left(\imath\, \mathcal{P}_\nu x_\nu\right).$$

Taking the trace of this relation in Hilbert space of large N QCD, and dividing by trace of unity, we would get the usual Wilson loop. We could avoid division by an infinite factor by taking the vacuum average instead.

$$\langle 0| \hat{T} \exp\left(\int_0^0 \nabla_\mu dx_\mu\right) |0\rangle = \frac{1}{N} \operatorname{tr} \hat{T} \exp\left(\int_0^0 A_\mu(x) dx_\mu\right) \tag{14}$$

This formal argument does not tell us how to compute this master field. The operator version of the Yang-Mills equations is not very helpful. It reads

$$[\nabla_\mu, [\nabla_\mu, \nabla_\nu]] = g_{eff}^2 \frac{\partial}{\partial \nabla_\mu}. \tag{15}$$

The operator $\frac{\partial}{\partial \nabla_\mu}$ in turn, is given by its commutator relations with ∇_μ.

Formally, inserting the left side of this equation inside the trace in the master field Anzatz (11) we would reproduce the correct loop equation provided the trace of the open loop is proportional to the delta function $\delta^d(x-y)$. The normalization would come out wrong, though, unless we include the divergent constant $\delta^d(0) = \Lambda^d$ in effective coupling constant

$$g_{eff}^2 = g_0^2 \Lambda^d \tag{16}$$

The Eguchi-Kawai reduction[6] was, in fact, a realization of this scheme. There, the large N matrix $\hat{A}_\mu$ represented the covariant derivative operator in above sense, except it was not a single matrix. Its diagonal components played the role of momenta $\hat{A}_\mu^{ii} = -\imath p_\mu^i$. These were classical, or, to be more precise, quenched[7] momenta.

The off-diagonal components were fluctuating with the weight $\exp\left(-\Lambda^d L_{YM}\right)$, corresponding to the Yang-Mills Lagrangean for the constant matrix field. The cutoff Λ explicitly entered these formulas. In effect, one could not take the local limit in the quenched EK model. Apparently something went wrong with implementation of the master field.

1.3 Noncommutative Probability Theory

Recently the large N methodology was enriched[10, 11] by using so called noncommutative probability theory[9]. In particular, in the last paper[11] the master field was constructed by similar method in the QCD2.

This amounts to the deformation of the commutation relation

$$a_\mu(p) a_\nu^\dagger(p') - q\, a_\nu^\dagger(p') a_\mu(p) = \delta_{\mu\nu} \delta^d(p-p') \tag{17}$$

Instead of the usual Bose commutation relations, with $q = 1$ one should take $q = 0$, which makes it so called Cuntz algebra. This algebra in the context of the planar diagram theory in QCD was first discovered long ago by Cvitanovic et.al.[8].

As noted in[11], this algebra corresponds to Boltzmann statistics. The origin of the Boltzmann statistics in the large N limit is clear: at large rank each component of the

matrix can be treated as unique object, neglecting the probability of the coincidences of indexes. The indexes could be dropped after this.

This interesting construction proves the concept, but we cannot be completely satisfied with the fact that the operator representation in this paper is gauge dependent. It used the fact that in the axial gauge there is no interaction in QCD2, so that the only deviation from the free field behaviour of the Wilson loop comes from the noncommutativity of the (Gaussian) gauge potential at various points in space.

This theory is purely kinematical, it does not make use of the large N dynamics. The Fock space is only a slight modification of the perturbative Fock space. The color confinement was not taken in consideration. There should exist a much tighter construction, built around the nonperturbative Fock space.

Also, the gauge dependence should be absent in the final formulation of the theory. This resembles the early attempts to solve the Quantum Gravity in 2D by means of perturbative Liouville theory. Taking the conformal gauge and using the methods of conformal field theory, one was forced to work in the space of free 2D particles, with the Bose operators $a_n, a_n^{\dagger}$.

However, as we know now, the Fock space of 2D Gravity is much smaller. It is described by the matrix models (or topological field theories), with only *discrete* states. If we now employ the same noncommutative probability theory, we could reinterpret the matrix models of 2D Gravity in terms of a single pair of operators $a, a^{\dagger}$.

To conclude this discussion, the ideas of the noncommutative probability are quite appealing, but implementation of these ideas in QCD is not yet finished. The correct nonperturbative Fock space in confining phase of QCD is yet to be found.

1.4 Position Master Field

In the present work, we follow the same general large N philosophy, but interpret the master field as the momentum operator $\hat{P}_\mu = \imath \hat{A}_\mu$ of the endpoint of the QCD string. With the second quantization, developed below, we rather construct the position operator $\hat{X}$. After that we shall reconstruct the momentum operator as well.

The advantage of the position operator is that (up to irrelevant global translational mode) we expect its spectrum to be bounded from above in confining phase.

The commutators $[\hat{X}_\mu, \hat{X}_\nu]$ describe the "area inside the Wilson loop". We could work with expectation values with products of such operators. As for the momentum operators, the high momentum region corresponds to small distance singularities of perturbative QCD. It makes the matrix elements of commutators of the Witten's master field infinite.

To rephrase it, the Wilson loop in coordinate space is singular. It cannot be expanded in powers of the size of the loop. There are the gluon exchange graphs, which lead to singularities at intersections and corners. The gluon exchange graphs for $W(C)$ scale as

powers of $|C|^{4-d}$.

In two dimensions the singularities reduce to existence of various topological sectors, with different Taylor expansions. Without self-intersections, there is a pure area law, with intersections there are polynomials of areas inside petals times exponentials[4].

In four dimensions there are logarithms, corresponding to the asymptotic freedom. It is not clear how to reproduce these properties by any constant large N matrix $\hat{A}_\mu$.

With the position operator $\hat{X}$, as we shall see, the correct coordinate dependence is built in our Anzatz (4). In particular, the δ function in the loop equation comes out automatically, without any assumptions about spectrum of $\hat{X}$. We escape the quenched EK model paradox of extra factor Λ^d in effective coupling constant.

2 Loop Equations

In this section we review and further develop the method of the loop equations in the large N QCD, first in coordinate space, then in momentum space.

2.1 Spatial Loops

We describe these loops by a periodic function of a proper time s

$$C : x_\mu(s) \; , x_\mu(s+1) = x_\mu(s). \tag{18}$$

The following Schwinger-Dyson equation is to be used

$$\begin{aligned} 0 = \int DA \operatorname{tr} \int_0^1 dt_1 x'_\nu(t_1)\, g_0^2 \frac{\delta}{\delta A_\nu(x(t_1))} \\ \exp\left(\int d^d x \frac{\operatorname{tr} F_{\mu\nu}^2}{4 g_0^2}\right) \hat{T} \exp\left(\int_0^1 ds A_\mu(x(s)) x'_\mu(s)\right). \end{aligned} \tag{19}$$

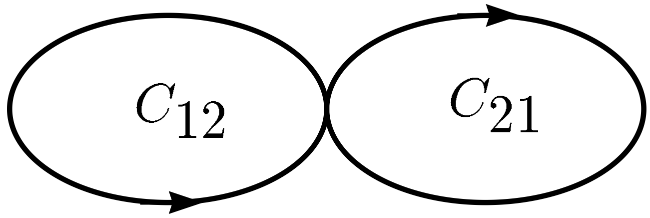
The variation of the first exponential yields the classical part of equation of motion,

$$[D_\mu \, F_{\mu\nu}] \, ,$$

whereas the second (ordered) exponential provides the commutator terms, present only for the self-intersecting loops. [2]

These terms involve $g_0^2 \oint dt_1 \oint dt_2 \delta^d \left(x(t_1) - x(t_2)\right) \delta_{\mu\nu}$ times the product of two ordered exponentials, corresponding to the two petals C_{12}, C_{21} of our loop C. The δ function ensures that both these petals are closed loops, as it is required by the gauge invariance.

[2]These self-intersections are responsible for all the "quantum corrections" to the classical equations of motion. Dropping them as "rare" or "exceptional" configurations would bring us back to the classical Yang-Mills theory.



The classical part can be represented[3, 14] as the functional Laplace (or Levy) operator

$$\hat{L} = \int_0^1 dt_1 \frac{\delta}{\delta x_\mu(t_1)} \int_{t_1-0}^{t_1+0} dt_2 \frac{\delta}{\delta x_\mu(t_2)} \tag{20}$$

acting on the loop functional

$$W[C] = \frac{1}{N} \left\langle \operatorname{tr} \hat{T} \exp\left(\int_0^1 ds A_\mu(x(s)) x'_\mu(s) \right) \right\rangle . \tag{21}$$

This Levy operator is the only parametric invariant second order operator, made of functional derivatives. It picks up the contact term in the second functional derivative

$$\begin{aligned} &\frac{\delta W[C]}{\delta x_\mu(t_1) \delta x_\mu(t_2)} \propto \\ &\delta(t_1 - t_2) x'_\nu(t_1) \frac{1}{N} \operatorname{tr} \hat{T} \left[D_\mu F_{\mu\nu}(x(t_1)) \right] \exp\left(\int_0^T A_\mu(x(s)) x'_\mu(s) \right) \\ &+ regular\, terms. \end{aligned} \tag{22}$$

On the right hand side, the average product of two loop functionals reduces at large N to the product of averages

$$\begin{aligned} &\frac{1}{N} \operatorname{tr} \hat{T} \exp\left(\int_{t_1}^{t_2} ds_1 A_\mu(x(s_1)) x'_\mu(s_1) \right) \\ &\frac{1}{N} \operatorname{tr} \hat{T} \exp\left(\int_{t_2}^{t_1+1} ds_2 A_\mu(x(s_2)) x'_\mu(s_2) \right) \\ &\xrightarrow{N \to \infty} W[C_{12}] W[C_{21}] \end{aligned} \tag{23}$$

We thus arrive at the conventional loop equation

$$\hat{L} W[C] = N g_0^2 \oint dt_1 x'_\mu(t_1) \int_{t_1^+}^{t_1^-} dt_2 x'_\mu(t_2) \delta^d \left(x(t_1) - x(t_2) \right) W[C_{12}] W[C_{21}], \tag{24}$$

where the second integral goes around the loop avoiding the first point t_1. Thus, the trivial self-intersection point at $t_2 = t_1$ is eliminated (see[3] for details).

Singularities in this equation should be properly understood. The double integral of the d dimensional δ function does not represent an ordinary number in more than two dimensions. This is a singular functional in the loop space, the only meaning of which could be provided by an integration with some weight function.

Otherwise we should mess up with the gauge invariant regularization of the delta function with inevitable ambiguity of the point splitting procedure. One such regularization is given by the lattice gauge theory, but the loss of the space symmetry makes the lattice loop equation both ugly and useless. Even the beautiful Eguchi-Kawai reduction could not save the lattice loop equation. It did not offer any alternatives to the lattice gauge theory.

2.2 Momentum Loops

Let us now forget about all the lattice artifacts and come back to continuum theory. The most natural thing to do with the delta function is to switch to momentum space. The parametric invariant functional Fourier transform of the Wilson loop is defined as follows

$$W[p] = \int DCW[C] \exp\left(-\imath \int_0^1 ds p_\mu(s) x'_\mu(s)\right), \tag{25}$$

where

$$DC = \left(\prod_{s=0}^{s<1} d^d x(s)\right) \delta^d(x(s_0)). \tag{26}$$

The δ function here fixes one arbitrary point $x(s_0) = 0$ at the loop, the choice of s_0 does not influence observables in virtue of translational invariance. Note, that the mass center fixing condition $\int_0^1 dsx(s) = 0$ is not parametric invariant, and would require the Jacobian. This is not so difficult to do, but our choice of one point fixing is simpler.

As it was noted in[12] this measure factorizes for self intersecting loops on the right hand side of the loop equation. The argument goes as follows (assuming $t_1 < t_2$, and choosing $s_0 = t_1$)

$$\begin{aligned} &\left(\prod_{s=0}^{s<1} d^d x(s)\right) \delta^d(x(t_1))\, \delta^d\left(x(t_1) - x(t_2)\right) = \\ &\left(\prod_{\tau=t_1}^{\tau<t_2} d^d x(\tau)\right) \delta^d(x(t_1)) \left(\prod_{\sigma=t_2}^{\sigma<t_1+1} d^d x(\sigma)\right) \delta^d(x(t_2)) = DC_{12}\, DC_{21}. \end{aligned} \tag{27}$$

The δ function for the loop self-intersection, $\delta^d\left(x(t_1) - x(t_2)\right)$ was combined with the point fixing δ function $\delta^d(x(t_1))$ to provide the point fixing δ functions for both petals. Note that this fits nicely large N factorization of the Wilson loops. For the abelian theory such factorization of the measure would be useless.

The functional Laplace operator on the left hand side of the loop equation transforms to the momentum space as follows

$$\int_0^1 dt_1 \frac{\delta}{\delta x_\mu(t_1)} \int_{t_1-0}^{t_1+0} dt_2 \frac{\delta}{\delta x_\mu(t_2)} \Rightarrow -\int_0^1 dt_1 p'_\mu(t_1) \int_{t_1-0}^{t_1+0} dt_2 p'_\mu(t_2). \tag{28}$$

As was discussed at length in[12], for continuous momentum loops this integral vanishes, but for the relevant loops with finite discontinuities, $\Delta p_k = p(t_k + 0) - p(t_k - 0)$, the

Laplace operator reduces to the sum of their squares $\sum_k (\Delta p_k)^2$. These discontinuities are induced by emission and absorption of gluons. We shall discuss this relation in more detail later.

In virtue of periodicity and symmetry this can also be written as the *ordered* integral

$$\begin{aligned}&\int_0^1 dt_1 p'_\mu(t_1) \int_{t_1-0}^{t_1+0} dt_2 p'_\mu(t_2) = \\ &-\int_0^1 dt_1 p'_\mu(t_1) \int_{t_1}^{t_1+1} dt_2 p'_\mu(t_2) = \\ &-\int_0^1 dt_1 p'_\mu(t_1) \int_{t_1}^{1} dt_2 p'_\mu(t_2) \\ &-\int_0^1 dt_1 p'_\mu(t_1) \int_{0}^{t_1} dt_2 p'_\mu(t_2) = \\ &-2\int_0^1 dt_2 p'_\mu(t_2) \int_{0}^{t_2} dt_1 p'_\mu(t_1)\end{aligned} \tag{29}$$

We should order the integration variables on the right as well, to avoid the trivial point $t_1 = t_2$ (one could trace this prescription back to the definition of the ordered exponent as an infinite product of the *linear* link factors $1 + A_\mu(x)dx_\mu$, which can be varied only once with respect to $A_\mu(x)$). After all this, we write the momentum loop equation as follows

$$W[p] \int_0^1 dt_2 p'_\mu(t_2) \int_0^{t_2} dt_1 p'_\mu(t_1) = \int_0^1 dt_2 \frac{\delta}{\delta p_\mu(t_2)} \int_0^{t_2} dt_1 \frac{\delta}{\delta p_\mu(t_1)} W\left[p_{12}\right] W\left[p_{21}\right]. \tag{30}$$

The parts p_{12}, p_{21} of original momentum loop are parametrized by smaller proper time intervals $T_{12} = t_2 - t_1, T_{21} = 1 + t_1 - t_2$. This can be rescaled to $(0,1)$ interval in virtue of the parametric invariance. With the manifestly invariant Anzatz (4), such rescaling is unnecessary.

Note that the momentum loops, unlike the coordinate loops, are not closed. There are inevitable discontinuities, in particular, both parts break at the matching point. Even if we would have assumed the initial loop $p(s)$ to be continuous, the W functionals on the right hand side would enter with the broken loops. Therefore, we have to allow such discontinuities on the left side as well, to get the closed equation.

The normalization of the momentum loop functional was chosen to absorb the bare coupling constant. The original normalization $W[C]_{x(.)=0} = 1$ results in the relation for the bare coupling constant

$$N g_0^2 = \int Dp W[p] \tag{31}$$

(with the same measure Dp as for the coordinate loop C).

We could treat this relation as a definition of the bare coupling constant. Here is where the ultraviolet divergences show up. We should cut these momenta, say, by the Gaussian cutoff $\exp\left(-\epsilon \int_0^1 ds p^2(s)\right)$. Surprisingly, such a brute force regularization preserves the gauge invariance, as our momenta were not shifted by a gauge field (see [12]).

This resembles the solution of $O(N)$ σ model in two dimensions, where the $\langle\, n_i n_i \,\rangle$ propagator was found to be the free one, with the mass given by the expectation value $\langle\, \sigma \,\rangle$ of Lagrange multiplier for the constraint $n_i n_i = 1$. This expectation value was then determined by the equation

$$1 = \langle\, n_i n_i \,\rangle = N g_0^2 \int \frac{d^2 k}{4\pi^2} \frac{1}{k^2 + \langle\, \sigma \,\rangle}$$

In both cases, one could regard this normalization condition as definition of unobservable bare coupling constant in terms of the ultraviolet cutoff. For the observables one could just leave the physical mass scale as the adjustment parameter.

The possibility to eliminate the bare coupling constant plus the scale invariance of the loop equation $W[p] \Rightarrow \alpha^{-4}\, W[\alpha\, p]$ makes this equation manifestly renormalizable. The physical mass scale cannot be found from the loop equation, it should rather be fitted to experiment. Then, just for the internal consistency check, the bare coupling constant can be *computed* as a function of the cutoff and the physical mass.

2.3 Planar Graphs in Momentum Loop Space

This is not to say, that the perturbation theory is unrecoverable from the momentum loop equation. The reader is invited to follow the direct derivation of the Faddeev-Popov diagrams from this equation[12, 13], starting with

$$W_0[p] = N g_0^2 \int \frac{d^d q}{(2\pi)^d} \delta[p(.) - q] \tag{32}$$

where

$$\delta[p(.) - q] \equiv \prod_{s=0}^{1} \delta^d(p(s) - q) \tag{33}$$

is the functional δ function.

In higher orders there are higher functional derivatives of the functional delta functions $\frac{\delta}{\delta p(\tau_m)} \delta[p(.) - q_l]$. The integration variables q_l serve as the momenta inside the loops of planar graphs (see[13] for details).

Say, in the second order we would have two momenta q_1, q_2 in two windows of the one-gluon graph on a disk. The difference $k = q_1 - q_2$ serves as the gluon momentum. There are two discontinuities $\pm k$ of the local momentum $p(s$ at the edge of the disk, therefore the operator $L(p) = 2k^2$ when applied to product $W_0[p_{12}] W_0[p_{21}]$ on the right side.

There are also some terms coming from the commutators of the functional derivatives $\frac{\delta}{\delta p(t_{1,2})}$ with $L(p)$. These terms drop in this order after integration over the loop. So, when inverted operator $\frac{1}{L(p)}$ commutes with these functional derivatives and reaches the product $W_0[p_{12}] W_0[p_{21}]$, it yields the gluon propagator $\frac{1}{2k^2}$.

In the third order we can no longer neglect the commutator terms. These terms are equivalent to the $[A_\mu, A_\nu]$ terms in a field strength. They produce the triple vertex in the planar graphs. It is amazing, that all these planar graphs with all their ghost and gauge fixing terms are generated by such a simple mechanism.

Each term is very singular, but as we argued in[12], these are spurious singularities, arising due to the nonperturbative origin of quark confinement. In fact, in a massive theory, there is no reason to expect singularities at zero momenta. The expansion coefficients in front of the momenta grow as $\exp\left(c/g_0^2\right)$ which is why there are singularities in the perturbation expansion.

If we assume that there is an expansion in powers of momenta, the natural mass scale M would emerge from the zero order term $W(0) = M^4$. The precise meaning of this expansion is established in the next section. We could use the scale invariance of the momentum loop equation to express the remaining terms in powers of $W(0)$.

2.4 Concluding Remarks

Another point to clarify. On the right hand side one should differentiate both W functionals, according to the identities[3]

$$\frac{\delta}{\delta p_\mu(t_i)} = \frac{1}{2}\frac{\delta}{\delta p_\mu(t_i+0)} + \frac{1}{2}\frac{\delta}{\delta p_\mu(t_i-0)}. \tag{34}$$

The resulting four terms reduce to two inequivalent ones

$$2W'W' + 2W''W. \tag{35}$$

One could introduce the area derivative in momentum loop space

$$\frac{\delta}{\delta p_\mu(t)} = p'_\nu(t)\frac{\delta}{\delta\sigma_{\mu\nu}(t)}, \tag{36}$$

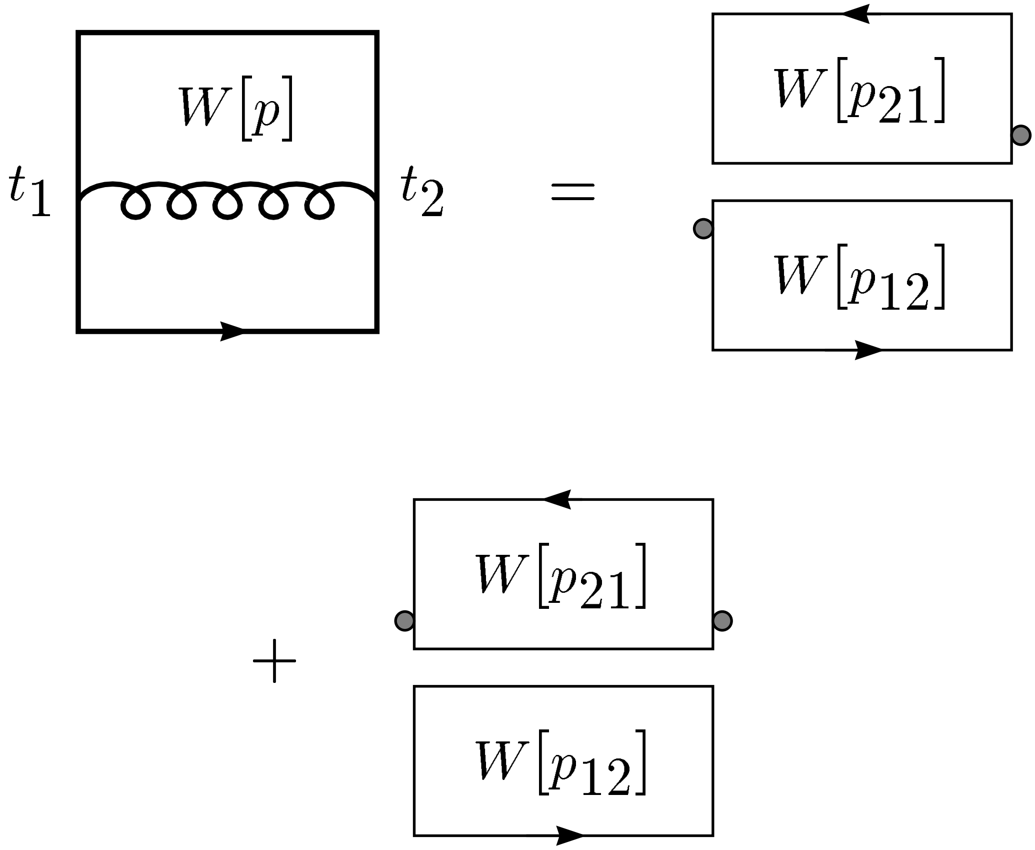
which would make the parametric invariance explicit.

So, finally, the momentum loop equation takes the form

$$\begin{aligned} W[p]\,\delta_{\alpha\beta}\int_0^1 dt_2 p'_\beta(t_2)\int_0^{t_2} dt_1 p'_\alpha(t_1) &= \int_0^1 dt_2 p'_\beta(t_2)\int_0^{t_2} dt_1 p'_\alpha(t_1) \\ \left[\frac{\delta W\left[p_{12}\right]}{\delta\sigma_{\mu\alpha}(t_1+0)}\frac{\delta W\left[p_{21}\right]}{\delta\sigma_{\mu\beta}(t_2+0)} + W[p_{12}]\frac{\delta^2 W\left[p_{21}\right]}{\delta\sigma_{\mu\alpha}(t_1-0)\delta\sigma_{\mu\beta}(t_2+0)}\right]&. \end{aligned} \tag{37}$$

We could depict it as follows

[3] Another way to arrive at the same point splitting prescription is to follow the Feynman's prescription of placing $A_\mu(x)$ in the middle of the interval dx in the loop product. Then $dx = x(t+\frac{1}{2}dt) - x(t-\frac{1}{2}dt)$ which leads to the half of the sum of the left and right velocities $dx = \frac{1}{2}x'(t+0)dt + \frac{1}{2}x'(t-0)dt$. The velocities x' then transform to the functional derivatives in p space.



It is tempting to assume that the integrands on the both sides coincide, but this is not necessarily true, as the perturbation theory shows. There are some terms, which vanish only after the closed loop integration. In the one gluon order these are the $k_\mu k_\nu$ terms in the gluon propagator. In general, these are the terms, related to the gauge fixing.

3 Models of the Momentum Loop

In this section we discuss various models and limits of the momentum loops to get used to this unusual object. In particular, we consider the QCD2, where the loops are classified by topology, and we also consider perturbative QCD in arbitrary dimension by WKB approximation.

3.1 "Exact" Solution

Let us first resolve the following paradox. There seems to be an "exact solution" of this equation of the form

$$W_?[p] = Z \exp\left(\frac{1}{2} G_{\mu\nu} \oint p_\mu dp_\nu\right). \tag{38}$$

Substituting this Anzatz into the right hand sidewe use the relations

$$\frac{\delta W_?[p]}{\delta \sigma_{\mu\nu}(s)} = G_{\mu\nu} W_?[p] \tag{39}$$

and note that the tensor area is additive

$$\oint_P p_\mu dp_\nu = \int_{P_{12}} p_\mu dp_\nu + \int_{P_{21}} p_\mu dp_\nu, \tag{40}$$

where P_{12} and P_{21} are parts of original loop P, closed with the straight lines. These straight lines cancel in the tensor area integral, as they enter with opposite orientation.

The exponentials trivially reproduce the left hand side, as for the factors in front, they yield

$$2Z^2 \int_0^1 dt_2 p'_\beta(t_2) \int_0^{t_2} dt_1 p'_\alpha(t_1) G_{\mu\alpha} G_{\mu\beta}. \tag{41}$$

This would agree with the left hand sideprovided

$$2Z G_{\mu\alpha} G_{\mu\beta} = \delta_{\alpha\beta}. \tag{42}$$

The rotations reduce the antisymmetric tensor $G_{\mu\nu}$ to the canonical form

$$G_{2k-1,2k} = -G_{2k,2k-1} = \pm \frac{1}{\sqrt{2Z}}. \tag{43}$$

with the rest of elements vanishing.

Clearly, something went wrong. The formal Fourier transformation would bring us back to the constant abelian field, which cannot solve QCD.

When we take a closer look at this Fourier transformation, we find out what went wrong. The Gaussian integral reduces to the solution of the classical equation

$$G_{\mu\nu} p'_\nu(s) = \imath\, x'_\nu(s), \tag{44}$$

which implies that $p(s)$ could have no discontinuities, since the loop $x(s)$ is continuous.

In order to reproduce the gluons we must satisfy the loop equations for the discontinuous momentum loops as well. In gluon perturbation theory, the Fourier integral involves the sum over number n of discontinuities and integrals over their locations s_i

$$W[C] = \sum_{n=0}^{\infty} \int ds_1 \dots ds_n \theta\left(s_1 < s_2 \dots < s_n\right) \tag{45}$$
$$\int (DP)_{s_1,\dots,s_n} W_{s_1,\dots,s_n}[p] \exp\left(\imath \oint p_\mu(t) dx_\mu(t)\right).$$

These amplitudes $W_n[p]$ are related to the the gluon $n-$ point functions $G_{\mu_1 \dots \mu_n}\left(k_1, \dots k_n\right)$ as follows

$$W_{s_1,\dots,s_n}[p] = \tag{46}$$
$$G_{\mu_1 \dots \mu_n}\left(k_1, \dots k_n\right) \delta\left(\sum k_i\right) \imath^n \frac{\delta}{\delta p_{\mu_1}(s_1)} \dots \frac{\delta}{\delta p_{\mu_n}(s_n)} \delta\left[p(\tau) - \sum k_i \theta(\tau - s_i)\right].$$

Clearly, the sum here could diverge. We expect the nonperturbative $W[p]$ to be a smooth *functional* of momenta (nevertheless these momenta could be non-smooth *functions* of their proper time variables).

In the case of the constant abelian field, performing the Fourier transformation from x to p space with n discontinuities $\Delta p(s_i) = k_i$ we would get

$$\begin{aligned}&\int Dx \exp\left(\frac{1}{2}\imath\, F_{\mu\nu} \oint x_\mu dx_\nu - \imath \oint x_\mu dp_\mu - \imath \sum x_\mu(s_i) k_i\right) \\ &\propto \exp\left(-\frac{1}{2}\imath\, (F^{-1})_{\mu\nu} \oint p_\mu dp_\nu\right) \prod_{i=1}^{n} \delta^d(k_i).\end{aligned} \tag{47}$$

The product of the delta functions comes from the integrations over the positions of x_i. This singularity arises because we have no $\dot{x}^2$ term in the exponential to damp spikes in $x(s)$.

So, the simple exponential in momentum space does not correspond to the constant abelian field. It would rather correspond to violation of the gauge invariance, as we could only obtain it by integrating over the discontinuous loops in coordinate space. Therefore we have to reject this formal solution.

3.2 Perturbative QCD

Let us now discuss the perturbative solution. The formal perturbation expansion for the momentum Wilson loop does not reflect its asymptotic behavior. To get a better description of the asymptotic freedom in momentum loop space, we should rather expand in the exponential of the path integral

$$W[p] \propto \int Dx \exp\left(\log(W(x)) - \imath \oint p'_\mu(s) x_\mu(s)\right), \tag{48}$$

near some (imaginary) saddle point $x(s) = C(s)$. It should satisfy the equation

$$\imath\, p'_\mu(s) = C'_\nu(s) \frac{\delta \log W(C)}{\delta \sigma_{\mu\nu}(s)}. \tag{49}$$

In the first nontrivial order we get an equation of the following form

$$\imath\, p'_\mu(s) = N g^2(C)\, C'_\nu(s) \oint \left(dC_\mu(t)\partial_\nu - dC_\nu(t)\partial_\mu\right) D\left(C(s) - C(t)\right). \tag{50}$$

Here the running coupling constant $Ng^2(C)$ corresponds to the scale of C and

$$D(x) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{\imath\, qx}}{q^2} \propto (x^2)^{1-\frac{1}{2}d}, \tag{51}$$

is the gluon propagator. We could take this classical equation as a *parametrization* of $p_C(s)$. We observe that $p_C \sim N g^2(C) |C|^{3-d}$. In four dimensions, for the small loop C,

the corresponding momenta are large. In two and three dimensions, large coordinate loops correspond to large momentum loops.

As for the powerlike divergency of the contour integral at $t \to s$ in three and four dimensions, it could be regularized by the principle value prescription $x^2 \to x^2 + \epsilon$. For the smooth nonintersecting loop C such integrals are finite.

The $\imath \oint C_\mu dp_C^\mu$ term reduces to -2 times the same one gluon exchange graph in virtue of the classical equation. So, we get the following large p asymptotics for $W[p]$ in the C-parametrization

$$\log W[p_C] = +\frac{1}{2} N g^2(C) \oint dC_\mu(s) \oint dC_\mu(t) D\left(C(s) - C(t)\right) \sim N g^2(C) |C|^{4-d}. \tag{52}$$

Unlike the formal perturbation theory, this expression is a well defined parametric invariant functional of $C(s)$. One could go further and compute the quadratic form for the second variations $\delta C(s)$ near the saddle point. This would give the pre-exponential factor in $W[P_C]$.

Note that in two dimensions there is no pre-exponential part for the non-intersecting loop. The first order of perturbation theory is exact for $\log W(C)$ in this case. The integrals can be done, and they reduce to the (absolute value of) the tensor area. Thus,

$$p_C^\mu(s) = \imath \; N g^2 \, \epsilon_{\mu\nu} \, C^\nu(s), \tag{53}$$

and

$$\log W[P] = -\frac{1}{2Ng^2} \left| \oint p_1 dp_2 \right|, \tag{54}$$

up to the nonlocal corrections due to self-intersecting C-loops in the path integral.

At present we do not know how to compute these corrections, but at large p_C the fluctuations $\delta C \sim \frac{1}{\sqrt{Ng^2}}$ are small compared to the classical loop $|C| \sim \frac{|p_C|}{Ng^2}$. Therefore, for large p we have the area law for $W[p]$. This is the general phenomenon, as one can see from the momentum loop equation. In fact, the above "derivation" of the tensor area law is valid for the large loops, where discontinuities and self-intersections are irrelevant.

3.3 Beyond Perturbation Theory: Random Vacuum Field

To get a rough idea what could be the momentum Wilson loop beyond perturbation theory, let us consider a simple "model" of QCD vacuum as random constant abelian field $F_{\mu\nu}$ with some probability distribution $P(F)$. As we discussed above, this is not a solution of the loop equations. Such model misses the hard gluon phenomena, responsible for asymptotic freedom, but it is the simplest model with confinement.

The Wilson loop would be

$$W(C) \sim \int P(F) dF \exp\left(\frac{1}{2} \imath \, F_{\mu\nu} \oint x_\mu \, dx_\nu\right), \tag{55}$$

and the corresponding momentum loop (for continuous $p(s)$)

$$W(p) \sim \int P(F) dF \sqrt{\det F} \, \exp\left(-\frac{1}{2}\imath \, F^{-1}_{\mu\nu} \, \oint p_\mu \, dp_\nu\right). \tag{56}$$

The pfaffian $\sqrt{\det F}$ comes as a result of computation of the functional determinant by the ζ regularization in Appendix A.

Both loop functionals could be expanded in Taylor series in this simple model. The Taylor expansion at small X exists, because there are no gluons, just the constant field. This is a wrong feature, to be modified. The Taylor expansion at small p exists, because the Wilson loop decreases fast enough in this model. This feature we expect to be present in the full nonperturbative QCD.

Note, that the Fourier integrals here are not the usual integrals with the Wiener measure, corresponding to Brownian motion. These are the parametric invariant integrals, without any cutoff for higher harmonics of the field $x(s)$ or $p(s)$. Such integrals can be defined as limits of the usual integrals, with discrete Fourier expansion at the parametric circle.

$$x(t) = \sum_{\infty}^{\infty} a_n e^{2\pi\imath \, nt}. \tag{57}$$

For the readers convenience, we consider these integrals in Appendix. We also compute the divergent pre-exponential factors with the ζ regularization.

The important thing to know about these integrals is that they exist. One does not have to break the parametric invariance by first adding the gauge fixing terms and then compensating that by some Jacobians. One could simply compute them with some invariant regularization, such as ζ, or Pauli-Villars regularization.

This also regularizes the wiggles, or backtracking. The Wilson loop does not change when a little spike with zero area (backtracking) is added to the loop. Such backtracking are hidden in the high harmonics a_n of the position field. The regularization of these wiggles leads to finite result in the constant field model. We expect the same in full nonperturbative QCD.

In this case, the position field could exist, unlike the usual master field. Our Anzatz (4) yields for the Taylor expansion terms

$$W_n(p) \propto \frac{1}{n!N} \operatorname{tr} \hat{T} \left(\oint_C dp_\mu(s) \hat{X}_\mu\right)^n = \tag{58}$$
$$\frac{1}{N} \int \dots \int \theta\left(s_1 < s_2 \dots < s_n\right) dp_{\mu_1}(s_1) \dots dp_{\mu_n}(s_n) \operatorname{tr}\left(\hat{X}_{\mu_1} \dots \hat{X}_{\mu_n}\right).$$

These terms, in fact, reduce to traces of products of commutators, as there is translational symmetry $X_\mu \Rightarrow X_\mu + 1 * a_\mu$.

Comparing the above path integral for these terms with the second quantized version we conclude that the string correlators

$$\frac{1}{n!} \int Dx \, \delta^d(x(s_0)) \, W(C) \, x_{\mu_1}(s_1) \dots x_{\mu_n}(s_n) \tag{59}$$

do not depend of proper times s_i, as long as these times are ordered. In general case these are combinations of the step functions, as they should be in parametric invariant (i.e. 1D topological) theory. Integrating these terms by parts with $dp_{\mu_1}(s_1)\dots dp_{\mu_n}(s_n)$ we would reproduce the commutator terms coming from derivatives of the step function $\theta\,(s_1 < s_2 \dots < s_n)$ in the ordered operator product representation.

These string correlations correspond to the Neumann boundary conditions. Here we do not fix the edge C of the "random surface", but rather let it fluctuate freely. This fixes the corresponding local momentum: $p_\mu = 0$ at the edge of the surface.

Adding the $\frac{1}{2}\alpha\dot{x}^2$ term at the edge would correspond to the term $\frac{1}{2}\frac{1}{\alpha}p^2$ in the Hamiltonian. We do have such terms in our physical Hamiltonian, of course, but for the purposes of computation of the simplest correlators we could set $\alpha \to 0$, which is equivalent to the boundary condition $p = 0$.

Our approach to the "QCD string" is similar to the matrix model approach to the 2D quantum gravity. We shall work directly in parametric invariant "sector" rather than fixing the gauge and then imposing constraints to come back to the physical sector.

4 Equations for Dual Amplitudes

The momentum loop equation is still ill defined. The meaning of the functional derivatives $\frac{\delta}{\delta p(t_i)}$ at the breaking points t_1, t_2 needs to be clarified. Also, the necessity to include momentum discontinuities makes the discrete Fourier expansion useless. Ideal thing would be some nonsingular equation with functions of finite number of variables instead of functionals.

4.1 Kinematics

Let us try to obtain the closed equations for the dual amplitudes

$$A\,(k_1, \dots k_n) = \frac{1}{N}\,\mathrm{tr}\,V(k_1)\dots V(k_n), \tag{60}$$

where

$$V(k) = \exp\left(\imath\, k_\mu X_\mu\right) \tag{61}$$

is the vertex operator, and momenta k_i add up to zero.

These traces are cyclic symmetric, of course. In addition there are relations between amplitudes with various number of points

$$A\,(\dots, \xi k, \eta k, \dots) = A\,(\dots, (\xi + \eta)k, \dots) \tag{62}$$

which do not have analogies in the particle theory. In particular, the two point function is trivial

$$A(k, -k) = A() = 1. \tag{63}$$

The first nontrivial one is the three point function $A(k, q-k, -q)$. Given the generic n point function, we automatically get all the previous functions by setting some momenta to zero.

The momentum loop represents the polygon with the sides k_i and vertices

$$p_i = \sum_{j=1}^{i} k_j,\; k_i = p_i - p_{i-1}, p_0 = 0. \tag{64}$$

In parameter space the sides are the discontinuities, they take infinitesimal time steps, $s_i < t < s_i + \epsilon$. This does not matter, in virtue of parametric invariance. The only thing which matters is the geometric shape of this polygon.

These amplitudes corresponds to $W[p]$ in the limit

$$p(s) \to \sum_{i=1}^{n} k_i \theta(s - s_i), \tag{65}$$

when there is no smooth part of momentum. Such limit is quite natural from the point of view of the string theory, but in perturbative QCD it would be singular.

Existence of such limit implies convergence of the path integrals with $W(C)$, which is violated in formal perturbation theory. In above constant field model such limit exist. The symplectic form becomes the quadratic form in momenta

$$F^{-1}_{\mu\nu} \oint p_\mu dp_\nu \Rightarrow \sum_{i=1}^{n} F^{-1}_{\mu\nu} k_i^\nu \sum_{j=1}^{i-1} k_j^\mu \tag{66}$$

Given the probability distribution function for the tensor F one could compute the corresponding amplitudes in this model.

4.2 Loop Equations

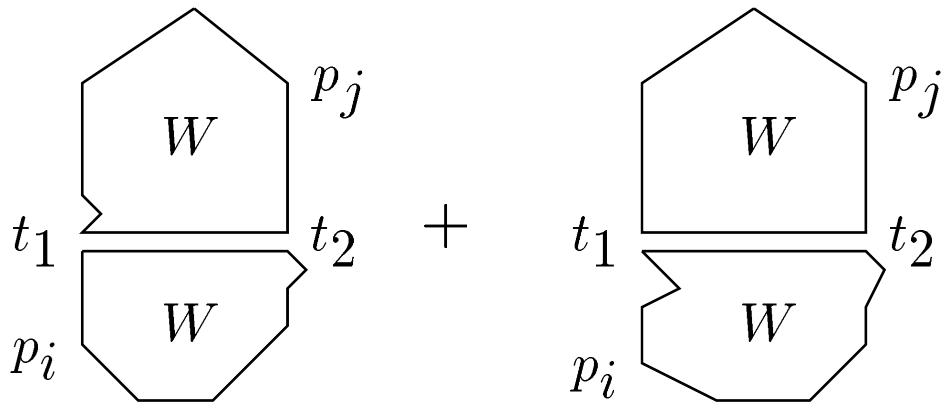
Let us turn to the loop equations in real QCD. We already know the limit of the left hand sideof the momentum loop equation. It is just

$$L(W)(p) \to \left(\sum_{i=1}^{n} k_i^2\right) A\left(k_1, \dots k_n\right). \tag{67}$$

This involves the sum of squares of lengths of the polygon.

The problem it to find the limit of the right hand sideof the loop equation. We have to compute it for the general Anzatz(4), and then tend the local momentum to the sum of the theta functions.

The double integral around our polygon breaks two of its sides



The extra discontinuity of momentum

$$\Delta = p(t_1) - p(t_2) \tag{68}$$

arising with opposite signs in the two momentum loops leads to extra vertex operators

$$\begin{aligned} W[p_{12}] &= \frac{1}{N} \operatorname{tr} \left(\hat{T} \exp \left(\imath \int_{t_1}^{t_2} d\, p_\mu \, \hat{X}_\mu \right) V(\Delta) \right), \\ W[p_{21}] &= \frac{1}{N} \operatorname{tr} \left(\hat{T} \exp \left(\imath \int_{t_2}^{t_1} d\, p_\mu \, \hat{X}_\mu \right) V(-\Delta) \right). \end{aligned} \tag{69}$$

These extra vertex operators correspond to internal sides, closing these two loops. They are not parts of external loop p, so they are not involved in the functional derivatives.

Let us now apply the functional derivative $\frac{\delta}{\delta p(.)}$ to our Anzatz. Start with the identity

$$-\imath \frac{\delta}{\delta p'_\nu(u)} \hat{T} \exp \left(\imath \int_s^t d\, p_\mu \, \hat{X}_\mu \right) = \hat{T} \exp \left(\imath \int_s^u d\, p_\mu \, \hat{X}_\mu \right) \hat{X}_\nu \, \hat{T} \exp \left(\imath \int_u^t d\, p_\mu \, \hat{X}_\mu \right), \tag{70}$$

and differentiate it in u. We find

$$\begin{aligned} -\imath \frac{\delta}{\delta p_\nu(u)} \hat{T} \exp \left(\imath \int_s^t d\, p_\mu \, \hat{X}_\mu \right) &= \imath \frac{\partial}{\partial u} \frac{\delta}{\delta p'_\nu(u)} \hat{T} \exp \left(\imath \int_s^t d\, p_\mu \, \hat{X}_\mu \right) \\ = p'_\alpha(u) \, \hat{T} \exp \left(\imath \int_s^u d\, p_\mu \, \hat{X}_\mu \right) [\hat{X}_\nu, \hat{X}_\alpha] \, \hat{T} \exp \left(\imath \int_u^t d\, p_\mu \, \hat{X}_\mu \right). \end{aligned} \tag{71}$$

These are momentum space analogs of the Mandelstam relation for the area derivative, with X_μ playing the role of the gauge field. These identities play the crucial role in our construction. Without them we would have only functional equations, for the general function $p(s)$.

When the momentum tends to the sum of the theta functions, the integrals over t_1, t_2 are concentrated inside the infinitesimal intervals $(s_i, s_i + \epsilon), (s_j, s_j + \epsilon)$, where derivative p' is present. Due to our principle value prescription they can never fall in the same interval. Inside such intervals the functional derivatives split the vertex operator

$$\delta V(k_i) \propto \int_{s_i}^{s_i+\epsilon} dp_\alpha(t) \left(V(k'_i) [\hat{X}_\nu, \hat{X}_\alpha] \otimes V(k''_i) + V(k'_i) \otimes [\hat{X}_\nu, \hat{X}_\alpha] V(k''_i) \right). \tag{72}$$

where

$$k_i' = p(t) - p(s_i), k_i'' = p(s_i + \epsilon) - p(t), k_i' + k_i'' = k_i. \tag{73}$$

The first factor in the tensor product goes in one trace, the second one goes in another trace, since the loops reconnect precisely at the variation point $t = t_1$ or $t = t_2$. The sum of two terms with commutators switching from one trace to another arises because our definition of velocity $x'(u)$ involves the sum of left and right derivatives.

Inside each infinitesimal interval $dp(t) = kd\xi, k' = \xi k, k'' = (1-\xi)k$, where $\xi = \frac{t-s_i}{\epsilon}$. Therefore we get

$$\delta V(k_i) \propto k_i^\alpha \int_0^1 d\xi \left(V(\xi k_i)[\hat{X}_\nu, \hat{X}_\alpha] \otimes V((1-\xi)k_i) + V(\xi k_i) \otimes [\hat{X}_\nu, \hat{X}_\alpha] V((1-\xi)k_i) \right). \tag{74}$$

The new objects here are the commutators. However, they can be obtained as a limit of the vertex operators

$$[\hat{X}_\mu, \hat{X}_\alpha]\, k_\alpha = 2 \left(\frac{\partial^2}{\partial q_\mu \partial \rho} V(\rho k) V(q - \rho k) V(-q) \right)_{\rho = q = 0} \tag{75}$$

Geometrically this corresponds to adding little triangle in the μ direction as shown at the above figure.

Now we have all the ingredients in place. We obtain the following equation

$$\begin{aligned}
&\left(\sum_{l=1}^{n} k_l^2 \right) A(k_1, \dots k_n) + 4 \sum_{i \neq j} \int_0^1 d\xi \int_0^1 d\eta \left(\frac{\partial^4}{\partial q_1^\mu \partial q_2^\mu \partial \rho_1 \partial \rho_2} \right)_{q_i = \rho_i = 0} \\
&A\left(\eta k_j, k_{j+1}, \dots, (1 - \xi + \rho_1) k_i, q_1 - \rho_1 k_i, -q_1, \Delta\right) \\
&A\left(\xi k_i, k_{i+1}, \dots, (1 - \eta + \rho_2) k_j, q_2 - \rho_2 k_j, -q_2, -\Delta\right) + \\
&A\left(-q_1, q_1 - \rho_1 k_i, (\xi + \rho_1) k_i, k_{i+1}, \dots, (1 - \eta + \rho_2) k_j, q_2 - \rho_2 k_j, -q_2, -\Delta\right) \\
&A\left(\eta k_j, k_{j+1}, \dots, (1 - \xi) k_i, \Delta\right) \\
&= 0.
\end{aligned} \tag{76}$$

where

$$\Delta = \xi k_i + \dots + (1 - \eta) k_j = -\left(\eta k_j + \dots + (1 - \xi) k_i\right) \tag{77}$$

is the transfer momentum.

These are exact equations for functions of finite number of variables rather than functionals. The main advantage of this equation over the old loop equation is absence of singularities. There are no delta functions here, the only k-dependence is polynomial.

5 Taylor Expansion at Small Momenta

One may expand these amplitudes in momenta (which should be possible in massive theory), and recurrently find the expansion coefficients. An important property of these equations is the fact that the same coefficients serve *all* the amplitudes, for arbitrary n.

5.1 Equations for Commutators

These expansion coefficients are related to the traces of products of the $\hat{X}$ operators. Such traces are universal tensors, composed from $\delta_{\mu\nu}$, independently of the number of k_i variables multiplying these tensors.

We shall denote by A_L the $O\left(k^L\right)$ terms in the amplitude, for generic momenta $k_1, \dots k_n$. In fact, this A_L would involve only combinations of L or less different momenta, so that remaining momenta are redundant. This allows us to find relations for these coefficients, taking only finite number of momenta.

The leading terms are quadratic in momenta. Matching these two quadratic forms gives the first nontrivial relation. The A term yields $\sum k_l^2$. In the remaining $A \otimes A$ terms only the product of the zeroth and fourth order terms A_4 contribute. The fourth order terms reduce to the trace of two commutators

$$\frac{1}{N} \mathrm{tr}\, [\hat{X}_\mu, \hat{X}_\alpha]\, [\hat{X}_\nu, \hat{X}_\beta] = a\, \left(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right). \tag{78}$$

We find the following sum rule[4]

$$2(d-1)a = 1 \tag{79}$$

In higher orders we get more relations, each time involving the higher derivative terms, i.e. higher commutators of position operator. In order to study these relations systematically, let us parametrize the conserved momenta as $k_i = p_i - p_{i-1}$ and rewrite the dual amplitude as follows

$$A\left(p_1 - p_n, \dots, p_n - p_{n-1}\right) = \frac{1}{N} \mathrm{tr}\, \left(S(p_1, p_n) \dots S(p_n, p_{n-1})\right) \tag{80}$$

with

$$S(k, q) = V(q)V(k-q)V(-k) = \hat{T} \exp\left(\imath \int_\triangle X_\mu dp_\mu\right) \tag{81}$$

where the ordered integral goes around the triangle made from three vectors $q, k-q, -k$. These triangles fill the polygon with vertices p_i and sides k_i.

At small momenta this formula reduces to

$$S(k, q) \to 1 + \frac{1}{2} k_\mu q_\nu\, [\hat{X}_\mu, \hat{X}_\nu] + \frac{\imath}{6}\left(k_\mu k_\nu q_\lambda - q_\mu q_\nu k_\lambda\right) [\hat{X}_\mu, [\hat{X}_\nu, \hat{X}_\lambda]] + \dots. \tag{82}$$

This is the nonabelian version of the Stokes theorem for the triangle. One can find higher terms here using the ordered integrals. From this representation it is obvious that the Taylor expansion starts from the quartic term, as the trace of commutator vanishes.

In higher orders we get the traces of polynomials to the set of traces of all possible commutators

$$\hat{X}_{\nu_1 \nu_2 \dots \nu_n} \equiv [\hat{X}_{\nu_1}, [\hat{X}_{\nu_2}, \dots \hat{X}_{\nu_n}] \dots]. \tag{83}$$

[4] It was first obtained by different method in collaboration with Ivan Kostov(unpublished).

The general translation invariant trace would involve products

$$T\left(\{\nu\}_n, \{\mu\}_m \ldots\right) = \operatorname{tr} \hat{X}_{\nu_1\nu_2\ldots\nu_n} \hat{X}_{\mu_1\mu_2\ldots\mu_m} \ldots . \tag{84}$$

These T-traces are not independent tensors. Those would be given by the usual traces

$$G\left(\mu_1 \ldots \mu_n\right) = \frac{1}{N} \operatorname{tr} \hat{X}_{\mu_1} \ldots \hat{X}_{\mu_n}, \tag{85}$$

except these traces do not exist because they are not invariant under translations $\delta\hat{X} = a * 1$.

5.2 Planar Connected Moments

The following parametrization proved to be convenient. Let us introduce the *planar connected* moments $W_{\mu_1\ldots\mu_n}$ defined recurrently as follows

$$\begin{aligned} &W_\mu = G(\mu); \\ &W_{\mu_1\ldots\mu_n} = G\left(\mu_1 \ldots \mu_n\right) - \sum_{planar} W_{\mu^{(1)}} \ldots W_{\mu^{(m)}}. \end{aligned} \tag{86}$$

Here the sum goes over all planar decompositions of the set $(\mu_1 \ldots \mu_n)$ into subsets $\left(\mu^{(1)}\right), \ldots \left(\mu^{(m)}\right)$. For example,

$$W_{\mu\nu} = G(\mu\nu) - G(\mu)G(\nu), \tag{87}$$

$$W_{\mu\nu\lambda} = G(\mu\nu\lambda) - G(\mu)G(\nu\lambda) - G(\nu)G(\lambda\mu) - G(\lambda)G(\mu\nu) + 2G(\mu)G(\nu)G(\lambda). \tag{88}$$

In the next section we shall use this representation to build the Fock space representation of the X operator.

The advantage of these $C-$ moments is their simple behaviour under translations. Only the first one gets translated

$$\delta\, W_\mu = a_\mu \tag{89}$$

while all the higher ones stay unchanged. This can be proven by induction. The variation of the $G-$ moments under infinitesimal translation reads

$$\delta\, G\left(\mu_1 \ldots \mu_n\right) = \sum_l a_{\mu_l} G\left(\mu_{l+1} \ldots \mu_{l-1}\right), \tag{90}$$

with the cyclic ordering implied. Straightforward planar graph counting gives the same for the variation of the planar representation

$$\begin{aligned} &G\left(\mu_1 \ldots \mu_n\right) = W_{\mu_1\ldots\mu_n} + \sum_{planar} W_{\mu^{(1)}} \ldots W_{\mu^{(m)}}, \\ &\delta\, G\left(\mu_1 \ldots \mu_n\right) = \sum_{planar} W_{\mu^{(1)}} \ldots \delta W_{\mu_l} W_{\mu^{(m)}} = \\ &\sum_l a_{\mu_l} \sum_{planar} W_{\mu^{(1)}} \ldots W_{\mu^{(m)}} = \\ &\sum_l a_{\mu_l} G\left(\mu_{l+1} \ldots \mu_{l-1}\right) \end{aligned} \tag{91}$$

In the next section we present a simpler proof based on operator representation.

So, we could fix the gauge by imposing

$$W_\mu = 0 \tag{92}$$

and parametrize the traces by the invariant coefficients in front of various terms in the planar connected moments. Those are constant tensors, composed from $\delta_{\mu_i \mu_j}$. To simplify the formulas we shall use the notation

$$\delta_{i,j} \equiv \delta_{\mu_i,\mu_j}. \tag{93}$$

This shall never lead to confusion, as all our indexes would be summed over at the end anyway.

Only cyclic symmetric tensors may appear. Up to the eighth order

$$\begin{aligned}
&W_{1,2} = \delta_{1,2}\, A, \\
&W_{1,2,3,4} = \delta_{1,3}\, \delta_{2,4}\, B + (\delta_{1,4}\, \delta_{2,3} + \delta_{1,2}\, \delta_{3,4})\; C, \\
&W_{1,2,3,4,5,6} = \delta_{1,4}\, \delta_{2,5}\, \delta_{3,6}\, D + \\
&(\delta_{1,4}\, \delta_{2,6}\, \delta_{3,5} + \delta_{1,5}\, \delta_{2,4}\, \delta_{3,6} + \delta_{1,3}\, \delta_{2,5}\, \delta_{4,6})\; E + \\
&(\delta_{1,6}\, \delta_{2,5}\, \delta_{3,4} + \delta_{1,2}\, \delta_{3,6}\, \delta_{4,5} + \delta_{1,4}\, \delta_{2,3}\, \delta_{5,6})\; F + \\
&(\delta_{1,5}\delta_{2,6}\delta_{3,4} + \delta_{1,6}\delta_{2,4}\delta_{3,5} + \delta_{1,3}\delta_{2,6}\delta_{4,5} + \delta_{1,5}\delta_{2,3}\delta_{4,6} + \delta_{1,2}\delta_{3,5}\delta_{4,6} + \delta_{1,3}\delta_{2,4}\delta_{5,6})\; G + \\
&(\delta_{1,6}\, \delta_{2,3}\, \delta_{4,5} + \delta_{1,2}\, \delta_{3,4}\, \delta_{5,6})\; H.
\end{aligned} \tag{94}$$

From the above solution for the fourth moment we find

$$B = C + A^2 + \frac{1}{2(d-1)}. \tag{95}$$

The coefficients A, C remain arbitrary.

5.3 Higher Orders

In higher orders, let us pick out the highest $O\left(k^L\right)$ order term in (76), moving the lower ones to the right

$$\begin{aligned}
&\sum_{i \neq j} k_i^\alpha\, k_j^\beta \int_0^1 d\xi \int_0^1 d\eta \\
&\operatorname{tr}\left([\hat{X}_\nu, \hat{X}_\alpha] V(\Delta) [\hat{X}_\nu, \hat{X}_\beta] V(\eta k_j) V(k_{j+1}) \dots V((1-\xi) k_i)\right)_{k^{L-2}} \\
&= F\left(A_{l<L-2}, k_j\right).
\end{aligned} \tag{96}$$

The left side involves (linearly!) the highest C coefficients, with $L+2$ indexes. The order of the coefficients on the right side is L and lower.

So, this chain of equations have triangular form, which is a great simplification. The number of equations we get in the $O\left(k^L\right)$ order is equal to the number of the L-th order cyclic symmetric invariants made from n momenta. The number of unknowns coincides with the number of cyclic symmetric tensors with $L+2$indexes, made from $\delta_{\mu\nu}$.

There are, in general, much more equations than unknowns, but some hidden symmetry which reduces this system to *just one equation*, at least up to the orders we computed.

Thus, all the $O\left(k^2\right)$ equations for any number n of legs reduce to (95). The $O\left(k^4\right)$ equations we studied, with up to 6 legs, all reduce to the following equation

$$3\,D = 2\,A^3 + A\,\left(\frac{7}{2(d-1)} - 4C\right) + E - 4\,F + 7\,G - H. \tag{97}$$

These equations were obtained by means of *Mathematica* . We generated all cyclic symmetric tensor structures recurrently, then introduced coefficient

in front and substituted into the operator expansion. The Cuntz algebra was implemented to compute the expectation values of the products of the $\hat{\mathcal{X}}$ operators.

We had to use some *Mathematica* tricks to increase efficiency in computations of the loop equations. In its present form the program could compute a couple more orders without much effort at the personal workstation. The program is included in the uuencoded hep-th files and could also be provided by e-mail upon request.

In the next order we found the equation, but it is too lengthy to write it down here. It is included in the *Mathematica* file.

We could not find the general formula for the number of parameters in d dimensions. The group theory must have these answers. Clearly, one cannot alternate more than d indexes, which restricts invariants made from $d+1$ or more $\delta_{\mu\nu}$ tensors.

In four dimensions this would start from ten indexes which we do not consider here. However, in two dimensions these restrictions are quite important.

5.4 Symplectic Symmetry in QCD2

There is, in fact more symmetry in two dimensions. The symplectic invariance of 2D QCD corresponds to the following transformations

$$\begin{aligned} \delta x_\mu(t) &= \alpha\epsilon_{\mu\nu}x_\nu(t), \\ \delta p_\mu(t) &= -\alpha\epsilon_{\mu\nu}p_\nu(t) \end{aligned} \tag{98}$$

One can readily check that these transformations preserve the symplectic form

$$\oint p_\mu dx_\mu,$$

as well as the tensor areas

$$\oint \epsilon_{\mu\nu}x_\mu dx_\nu, \oint \epsilon_{\mu\nu}p_\mu dp_\nu.$$

We already saw this symmetry in the WKB approximation at large p when the momentum loop was a function of the tensor area in momentum space.

These transformations lead to the symmetry of our equations in 2D

$$\delta \hat{X}_\mu = \alpha \epsilon_{\mu\nu} \hat{X}_\nu. \tag{99}$$

As a consequence of this symmetry, one should use only the $\epsilon_{\mu\nu}$ symbols in constructing the expectation values of products of commutators of $\hat{X}$. The parity implies that the ϵ symbols come in pairs, therefore only the multiples of four $\hat{X}$ operators could have nonzero traces. The lowest commutator $\hat{X}_{\mu\nu} = [\hat{X}_\mu, \hat{X}_\nu]$ is symplectic invariant, but the higher ones, in general, are not. This leads to extra restrictions on the invariant structures, to be used in the solution.

Up to the eighth order the symplectic invariant solution in two dimensions reads

$$A = 0,\ B = \frac{1}{3},\ C = -\frac{1}{6} \dots \tag{100}$$

It corresponds to

$$W_{\alpha\beta\mu\nu} = \frac{1}{6} \left(\epsilon_{\alpha\beta}\epsilon_{\mu\nu} + \epsilon_{\beta\mu}\epsilon_{\nu\alpha}\right). \tag{101}$$

5.5 Higher Dimensions

In higher dimensions it is tempting to try something similar. Due to the Z_4 symmetry

$$p_\mu(s) \Rightarrow \omega p_\mu(s),\ \omega^4 = 1, \tag{102}$$

of the momentum loop equation there is always a solution with $W_{1,\dots,n} = 0$ unless $n = 0 mod 4$, so that the Taylor expansion goes in p^4.

This is not sufficient to eliminate the higher invariants in $C-$ tensors. Say, in the fourth order the second invariant C would still be present

$$B = C + \frac{1}{2(d-1)}. \tag{103}$$

Similar terms would exist in higher orders. We do not have symplectic symmetry here to reduce invariants, so what do we do?

In principle, we should fix the unknown coefficients by matching with perturbative QCD at large momenta. This is easier said than done. As a model, we could try to build the general terms from commutators $\hat{X}_{\mu\nu}$ in arbitrary dimension. Say, in the fourth order this would correspond to

$$C = -\frac{1}{2} B, \tag{104}$$

so that the $C-$ tensor would be equivalent to the double commutator

$$W_{1,2,3,4} = \frac{1}{2} B \left(2\, \delta_{1,3}\, \delta_{2,4} - \delta_{1,4}\, \delta_{2,3} - \delta_{1,2}\, \delta_{3,4}\right). \tag{105}$$

Solving the loop equation we find

$$B = \frac{1}{3(d-1)}. \tag{106}$$

In four dimensions we could get some number of Taylor terms analytically. Then, the Pade approximation could be used to estimate the spectrum. One could also try to compare some string models with these relations.

In Appendix B we discuss the possible numerical solution of these equations. We postpone the numerics till the better understanding of analytic properties of the momentum loop equation.

6 Operator Representation

The equations of the previous section can be used to recurrently determine the planar connected moments $W_{\mu_1 \dots \mu_n}$.

Suppose we solved this algebraic problem. The natural next question is how to reconstruct the matrix $\hat{X}$ from its moments. In the large N limit this problem can be explicitly solved by means of the noncommutative probability theory.

6.1 Noncommutative Probability Theory

The noncommutative probability theory in its initial form[9] covered only so called free noncommutative variables, corresponding to the factorized measure. The theory of interacting noncommutative variables which we use here, was developed in the recent work[11], where it was applied to the matrix models and QCD2. Some elements of this theory were discovered back in the eighties [8] in the context of the planar graph Schwinger-Dyson equations.

Let us introduce auxiliary operators a_μ and $a_\mu^\dagger$ satisfying the Cuntz algebra

$$a_\mu a_\nu^\dagger = \delta_{\mu\nu} \tag{107}$$

which is a limit of the $q-$ deformed oscillator algebra

$$a_\mu a_\nu^\dagger - q\, a_\nu^\dagger a_\mu = \delta_{\mu\nu} \tag{108}$$

when $q \to 0$. These operators correspond to the source $j = a^\dagger$ and noncommutative derivative $\frac{\delta}{\delta j} = a$ introduced by Cvitanovic et. al.[8].

We could explicitly build these operators as the limit of the large N matrix $a_\mu^\dagger$ and its matrix derivative $a_\mu = \frac{1}{N}\frac{\partial}{\partial a_\mu^\dagger}$. The trick is that we shall keep all the traces of products of $a^\dagger$ *finite* in the large N limit, so that the moments vanish except the zeroth

$$\frac{1}{N} \operatorname{tr} a_{\mu_1}^\dagger \dots a_{\mu_n}^\dagger = \delta_{n0}. \tag{109}$$

The matrix derivative is the $N = \infty$ counterpart of the matrix of the derivatives with respect to the matrix elements of a_μ

$$(a_\mu)^{ij} = \frac{1}{N} \frac{\partial}{\partial \left(a^\dagger_\mu\right)^{ji}}. \tag{110}$$

The transposition of indexes ij is needed to obtain the simple matrix multiplication rules

$$\begin{aligned}&\left(a_\mu a^\dagger_{\mu_1} \dots a^\dagger_{\mu_n}\right)_{ik} = (a_\mu)^{ij} \left(a^\dagger_{\mu_1} \dots a^\dagger_{\mu_n}\right)_{jk} = \qquad (111)\\ &\delta_{\mu\mu_1} \left(a^\dagger_{\mu_2} \dots a^\dagger_{\mu_n}\right)_{ik} + \frac{1}{N} \sum_{l=2}^{n} \delta_{\mu\mu_l} \operatorname{tr} \left(a^\dagger_{\mu_1} \dots a^\dagger_{\mu_{l-1}}\right) \left(a^\dagger_{\mu_{l+1}} \dots a^\dagger_{\mu_n}\right)_{ik} \rightarrow \delta_{\mu\mu_1} \left(a^\dagger_{\mu_2} \dots a^\dagger_{\mu_n}\right)_{ik}\end{aligned}$$

In the first term the factor of N in definition of the matrix derivative cancelled with the trace $\delta_{ii} = N$. The remaining terms all contain the vanishing moments.

6.2 Operator Expansion for the Position Operator

Let us define the following *operator expansion*

$$\hat{\mathcal{X}}_\mu = W_\mu * 1 + a_\mu + \sum W_{\mu\nu_1\dots\nu_n} a^\dagger_{\nu_n} \dots a^\dagger_{\nu_1} \tag{112}$$

where the sum goes over all planar connected moments C with all tensor indexes. The first term is a c-number. It takes care of the possible vacuum average of the position operator. It drops in the dual amplitudes, as it commutes with the rest of terms and corresponding exponent can be factored out of the vertex operator

$$\hat{\mathcal{V}}(k) = \exp\left(\imath\, k_\mu \hat{\mathcal{X}}_\mu\right) \rightarrow \exp\left(\imath\, k_\mu W_\mu\right) \hat{\mathcal{V}}(k). \tag{113}$$

These phase factors all cancel due to momentum conservation, which allows us to drop the $W_\mu-$ term from the position operator.

With this representation it is obvious that the planar connected moments $W_{\dots}$ are all translational invariant, except for the first one, W_μ. The translation of the c-number term provides the correct shift of the position operator, and there is no way the variation of higher terms could have compensated each other, as they enter in front of different products of the $a^\dagger_\mu$ operators.

Consider the expectation values of the products of such operators with respect to the "ground state" $|0\rangle$ annihilated by a_μ

$$a_\mu |0\rangle = 0,\ \langle 0| a^\dagger_\mu = 0. \tag{114}$$

With our matrix derivative representation of these operators

$$|0\rangle = 1,\ \langle 0| = \delta(a^\dagger); \tag{115}$$

so that

$$\langle 0|\, a_{\mu_1} \dots a_{\mu_n} F(a^\dagger)\, |0\rangle = \frac{1}{N}\frac{\partial}{\partial a^\dagger_{\mu_1}} \dots \frac{1}{N}\frac{\partial}{\partial a^\dagger_{\mu_n}} F(a^\dagger \to 0) \tag{116}$$

Clearly, only the terms with all a and $a^\dagger$ mutually cancelled will remain. This gives the usual decomposition of the moments into planar connected ones

$$\langle 0|\, \hat{\mathcal{X}}_{\mu_1} \dots \hat{\mathcal{X}}_{\mu_1}\, |0\rangle = W_{\mu_1\dots\mu_n} + \sum W_{\dots} W_{\dots} + \dots \tag{117}$$

The details of this beautiful construction for matrix models and 2d QCD can be found in [11].

In our case, the coefficients of the operator expansion are governed by the loop equations. Up to the sixth order

$$\hat{\mathcal{X}}_\mu = a_\mu + A\, a^\dagger_\mu + B\, a^\dagger_\nu\, a^\dagger_\mu\, a^\dagger_\nu + C\, \left(a^\dagger_\mu\, a^\dagger_\mu\, a^\dagger_\nu + a^\dagger_\nu\, a^\dagger_\mu\, a^\dagger_\mu\right) + \dots \tag{118}$$

with the coefficients $A, B, C, \dots$ satisfying above equations (95),(97).

The simplest solution, discussed in the previous section, reads,

$$\hat{\mathcal{X}}_\mu = a_\mu + \frac{1}{6(d-1)} \left[a^\dagger_\nu \left[a^\dagger_\mu\, a^\dagger_\nu\right]\right] + O\left((a^\dagger)^7\right) \tag{119}$$

What could be the use of this expansion? Perhaps, some truncation procedure using some trial functions can be found. Or else, one can always use the analytic methods, by extrapolation of expansion for the physical observables by the Padé technique. We do not know all the expansion parameters, they are to be fixed by matching with perturbative QCD at large momenta.

6.3 Perturbative vs Nonperturbative Fock Space

It is instructive to compare this construction with that of [11]. We have here (at arbitrary dimension of space) much smaller Fock space than the one found by Gopakumar and Gross for QCD2. There, the operators $a, a^\dagger$ had the momenta as their arguments, which made the Fock space similar to the space of the free gluons (except for the Boltzmann statistics instead of the Bose statistics of the usual free gluon theory).

The present construction has much less degrees of freedom, in fact, no continuous degrees of freedom at all. We simply have d pairs $a_\mu, a^\dagger_\mu$ which is like the Fock space of a single Boltzmann oscillator. The states are spanned by all $d-$ letter words

$$|\mu_1 \mu_2 \dots \mu_n\rangle = a^\dagger_{\mu_1} a^\dagger_{\mu_2} \dots a^\dagger_{\mu_n}\, |0\rangle\, . \tag{120}$$

This tremendous reduction of degrees of freedom (very much like the one in Eguchi-Kawai model, but in continuum theory) takes place due to implied confinement.

It is due to confinement that we are able to expand amplitudes in local momenta, so that only the Taylor coefficients need to be described. The Taylor expansion is much simpler that the Fourier expansion, where the basic set of functions are labelled by continuous momenta.

Another simplification came from unbroken parametric invariance. The general Taylor expansion of the loop functional would involve unknown coefficient functions, which would result in the loop operators $a_\mu(s)$, like those of the conventional string theory. Expanding those in discrete Fourier expansion on a unit circle we would get the usual infinite set of the string oscillators $a_{\mu,n}$.

So, why is our Fock space smaller that that of the string theory? This is because we did not fix the parametric gauge either. Should we break the parametric invariance by adding the conventional $\dot{x}_\mu^2$ term in the exponential in the definition of the path integral, (i.e. use the Wiener measure instead of the flat one we use here), we would have all these problems back again.

However, as we learned from the recent success of the matrix models, the quantum geometry problems can be solved by purely algebraic and combinatorial methods, without any breaking of parametric invariance. The (perturbative) Fock space of the Liouville theory in conformal gauge involves continuum fields, but we know that there is a different picture, with only discrete degrees of freedom.

This picture is the matrix model quantized by means of the noncommutative probability theory. The Fock space in this picture involves only one Cuntz pair $a, a^\dagger$. The perturbative Fock space was too big as it turned out after exact solution. After that, the Liouville theory Fock space was also reduced, by methods of the topological field theory.

So, we expect the same thing in the large N QCD. The nonperturbative Fock space must be much smaller than the perturbative one due to the color confinement. The operators we build here are discrete because we are working in purported confining phase of QCD, where the motion is finite, so that most of the perturbative gluon degrees of freedom are eliminated.

By the same reason we switched from the original gauge field (the Witten's master field) to the conjugate coordinate operator. The master field was unbounded even in confining phase due to the hard gluons. The coordinate field is bounded in confining phase, which is infinitely simpler to describe in our picture than the free motion.

In fact, without confinement, our methods would be completely pointless. The description of the plane wave of unconfined gluon in terms of Taylor coefficients would be very inefficient.

The proof of quark confinement within such approach cannot be achieved, all we can say is that exact equations for the dual amplitudes allow for a systematic expansion in local momenta, which can be interpreted in operator terms.

Note however, that these same equations, being iterated from perturbative vacuum

instead, generate all the planar graphs of perturbative QCD. The equations being non-linear, we cannot say whether our Taylor expansion describes the same solution, which matches perturbative QCD.

7 Hadron Spectrum

In this section we derive the wave equations for the mesons and glueballs. These equations correspond to a different physics: stringy states described by a long timelike Wilson loop in meson case, and small fluctuations of the Wilson loop in the glueball case. Still, in both cases, the wave equations can be explicitly written down in terms of the position operator.

7.1 Mesons

The equations of the previous section are supposed to fix $\hat{X}$ up to the overall gauge transformation $\hat{X}_\mu \Rightarrow S^{-1}\hat{X}_\mu S$. Let us assume we found the solution for $\hat{X}$, numerically or analytically. What can be said about the particle spectrum? Let us study the meson sector first.

Consider the limit, when the loop becomes an infinite strip. This corresponds to the propagation of the meson made from quark and antiquaqrk. There are two position operators $\hat{X}_\mu, \bar{\hat{X}}_\mu$, and the corresponding loop function can be regarded as tensor product of the two ordered exponentials

$$\Psi\left(T, \hat{X}, \bar{\hat{X}}\right) \propto \int Dp(.) \int D\bar{p}(.)\, \hat{T} \exp\left(\int_{-\infty}^{T} dt \left(\imath\, \hat{X}_\mu \frac{dp_\mu}{dt} - H\right) \oplus \left(\imath\, \bar{\hat{X}}_\mu \frac{d\bar{p}_\mu}{dt} - \bar{H}\right)\right) \tag{121}$$

with the above Dirac Hamiltonians (10).

Let us shift the time $T \Rightarrow T+dT$ and try to get the Schrödinger equation (in imaginary time). Introducing the vectors $q_\mu = p_\mu(T+dT) - p_\mu(T)$ we get the vertex operators

$$\hat{T} \exp\left(\imath \int_T^{T+dT} \hat{X}_\mu dp_\mu(t)\right) \to \exp\left(\imath\, q_\mu \hat{X}_\mu\right) = V(q). \tag{122}$$

Now, expanding the rest of the factors in dT

$$\exp\left(-dT\left(m + \imath\,\gamma_\mu p_\mu(T+dT)\right)\right) = \exp\left(-dT\left(\hat{H} + \imath\,\gamma_\mu q_\mu\right)\right) \to 1 - dT\left(\hat{H} + \imath\,\gamma_\mu q_\mu\right) \tag{123}$$

and comparing the linear terms, we get the following

$$-\dot{\Psi}\left(T, \hat{X}, \bar{\hat{X}}\right) = \left(\left(H + \gamma_\mu \hat{A}_\mu\right) \oplus \left(\bar{H} + \bar{\gamma}_\mu \bar{\hat{A}}_\mu\right)\right) \Psi\left(T, \hat{X}, \bar{\hat{X}}\right), \tag{124}$$

where

$$\hat{A}_\mu = \imath \int \frac{d^d q}{(2\pi)^d} V(q) q_\mu = \partial_\mu \delta(\hat{X}). \tag{125}$$

We also get some terms with the quark mass corrections, which we absorb into the bare mass m.

The $\hat{A}_\mu$ term is the net contribution of the "QCD string". This is the natural generalization of the derivative operator in the Dirac equation. Incidentally, this is what we get for the Witten's master field.

For a finite matrix $\hat{X}$ this term would be a sum of the (gradients of the) delta functions, whereas in the large N limit it is related to the density of eigenvalues at small $\hat{X}$. The singularities at small loops would correspond to the divergent integrals at small eigenvalues.

There are d different sets of eigenvalues κ_μ and the same number of different eigenvectors $\hat{\Omega}_\mu$, as the matrices $\hat{X}_\mu = \hat{\Omega}_\mu \kappa_\mu \hat{\Omega}_\mu^\dagger$ do not commute. Therefore, the angular variables, corresponding to the eigenvectors of each $\hat{X}_\mu$ would not decouple here.

With the operator language described in the previous section, the gradient of the delta function can be defined by its matrix elements with some trial functions in the Fock space of the operators $a_\mu^\dagger$.

Once we know the master field, the above Schrödinger equation is quite explicit. The states with fixed total momentum

$$P^{tot} = P^{string} + p + \bar{p} \tag{126}$$

are selected by condition

$$\imath P_\mu^{string} \Psi\left(T, \hat{X}, \bar{\hat{X}}\right) = \left(\operatorname{tr}\left(\frac{\partial}{\partial \hat{X}_\mu}\right) + \operatorname{tr}\left(\frac{\partial}{\partial \bar{\hat{X}}_\mu}\right)\right) \Psi\left(T, \hat{X}, \bar{\hat{X}}\right), \tag{127}$$

corresponding to total translation. The mass spectrum is given by stationary solutions, i.e.

$$\left(\left(H + \gamma_\mu \hat{A}_\mu\right) \oplus \left(\bar{H} + \bar{\gamma}_\mu \bar{\hat{A}}_\mu\right)\right) \Psi\left(\hat{X}, \bar{\hat{X}}\right) = 0. \tag{128}$$

The appropriate trial states are superpositions of the vertex operators

$$\begin{aligned} &\Psi_{k_1, \dots k_n, \bar{k}_1, \dots \bar{k}_m}\left(\hat{X}, \bar{\hat{X}}\right) = V(k_1) \dots V(k_n) \otimes \bar{V}\left(\bar{k}_1\right) \dots \bar{V}\left(\bar{k}_m\right), \\ &\sum k_i + \sum \bar{k}_j = P^{string}. \end{aligned} \tag{129}$$

The above wave operator acts linearly on the space of such states. Note, that we are not considering the dual amplitudes now. The quark degrees of freedom are present in this hamiltonian. Still, the vertex operators can be used as the basis for the wave equation. The computational aspects are beyond our present scope.

7.2 Glueballs

In the glueball sector the situation is different. The glueball wave function $G(C)$ can be obtained of as variation of the Wilson loop $G(C) = \delta W(C)$, in the same way, as the

variations of the Higgs condensate describe the physical excitations in the phase with "broken" gauge symmetry.

This relation between $G(C)$ and $W(C)$ can be formally derived from the loop equation for the generalized loop average

$$W_Q(C) = \left\langle Q \operatorname{tr} \left(\hat{T} \exp \left(\oint_C A_\mu dx_\mu \right) \right) \right\rangle \tag{130}$$

where Q is some invariant operator with the fixed total momentum,

$$Q = \int d^d y \exp\left(-\imath\, ky\right) \operatorname{tr}\, \left(P\left(F_{\mu\nu}, \nabla_\alpha F_{\beta\gamma}, \ldots\right)\right) \tag{131}$$

One could think of this Q as an infinitesimal loop, moved around with the plane wave weight $\exp\left(-\imath\, ky\right)$.

The functional $W_Q(C)$ is not translation invariant, it is rather the momentum eigenstate

$$\oint ds \frac{\delta}{\delta x_\mu(s)} W_Q(C) = \imath\, k_\mu W_Q(C). \tag{132}$$

as it follows from the translation $y \to y + a$ in above integral.

The Levy operator still yields the Yang-Mills equation inside the trace. In virtue of the Schwinger-Dyson equations this leads to the usual loop splitting terms plus the contact terms, which we ignore, as we are interested in the residue in the pole at the glueball spectrum $k^2 = -m^2$. The loop equation for this residue $G_Q(C)$ reads

$$L(G_Q)(C) = R(G_Q, W)(C) + R(W, G_Q)(C) \tag{133}$$

We used here the factorization theorem, keeping in mind that for finite momentum k the expectation value of Q vanishes. Hence, the leading term in the average product of the three invariant operators reads

$$\begin{aligned}
&\left\langle Q \operatorname{tr} \left(\hat{T} \exp \left(\oint_{C_{xy}} A_\mu dx_\mu \right) \right) \operatorname{tr} \left(\hat{T} \exp \left(\oint_{C_{yx}} A_\mu dx_\mu \right) \right) \right\rangle \to \\
&\left\langle Q \operatorname{tr} \left(\hat{T} \exp \left(\oint_{C_{xy}} A_\mu dx_\mu \right) \right) \right\rangle \left\langle \operatorname{tr} \left(\hat{T} \exp \left(\oint_{C_{yx}} A_\mu dx_\mu \right) \right) \right\rangle + \\
&\left\langle Q \operatorname{tr} \left(\hat{T} \exp \left(\oint_{C_{yx}} A_\mu dx_\mu \right) \right) \right\rangle \left\langle \operatorname{tr} \left(\hat{T} \exp \left(\oint_{C_{xy}} A_\mu dx_\mu \right) \right) \right\rangle
\end{aligned} \tag{134}$$

Now, the loop equation (133) would be satisfied for the *linear variation*

$$G_Q(C) = \delta W(C, \hat{X}) \equiv W\left(C, \hat{X} + \delta\hat{X}\right) - W(C, \hat{X}), \tag{135}$$

along the nonlinear loop equation. In other words, if the nonlinear loop equation is satisfied, when the position field $\hat{X}$ is shifted by $\delta\hat{X}$, then the variation of $W(C, \hat{X})$ satisfies the linearized loop equation (133).

Such perturbation corresponds to the variation of the vertex operator

$$\delta V(k) = \int_0^1 d\lambda\, V(\lambda k)\, \imath\, k_\mu\, \delta \hat{X}_\mu\, V((1-\lambda)k). \tag{136}$$

with $\delta \hat{X}_\mu$ represented by the operator expansion

$$\delta \hat{\mathcal{X}}_\mu = \delta W_\mu + \sum \delta W_{\mu\mu_1\ldots\mu_n} a^\dagger_{\mu_n} \ldots a^\dagger_{\mu_1}, \tag{137}$$

The set of equations for the perturbed amplitudes

$$\delta A(k_1, \ldots k_n) = \sum_{l=1}^{n} \imath\, k_l^\mu \int_0^1 d\lambda\, \mathrm{tr}\, \delta \hat{X}_\mu\, V((1-\lambda)k_l)\, V(k_{l+1}) \ldots V(k_{l-1})\, V(\lambda k_l) \tag{138}$$

would now provide the linear relations for the coefficients $\delta W_{\ldots}$. This variation should be the momentum eigenstate, i.e.

$$\delta A(k_1, \ldots k_n) \propto \delta^d(k - \sum k_i) \tag{139}$$

after which we get linear homogeneous system of equations, involving k_μ as a parameter. The solvability conditions should provide the spectral equation for Q. In virtue of the space symmetry this would quantize the square $k_\mu^2 = -M^2$.

8 Conclusion

This is as close as we could get to the QCD string. We built the model of d large N matrices $\hat{X}_\mu$ representing position operators of the open string endpoint. The infinite set of algebraic relations for the matrix elements of of $\hat{X}$ was obtained. The Fock space representation of these infinite matrices in terms of the operator expansion was found.

The expansion coefficients are recurrently related to each other, which allows them to be explicitly computed in QCD2, and strongly reduced in general case. The problem of elimination of unknown parameters in the Taylor expansion in general case remains opened.

There is nothing like the functional integral over random surfaces, which is not so bad after all. Such integrals have problems even in critical dimension of space, where there are tachyons. As for the noncritical strings, they do not yet exist. Hopefully we would see such string theories in future[16], then they would have a chance to be equivalent to QCD.

The advantage of the present "string" theory is that it is manifestly parametric invariant. We do not reduce the larger state to the parametric invariant sector by Virasoro constraints, we directly work in physical sector. In this sense we are building our "string model" as one dimensional topological field theory.

We bypassed the hard unsolved problem of computation of the Wilson loops in coordinate space. Not only these loops are hard to compute, they are not defined as numbers in a usual sense, but rather as *distributions* in loop space. The momentum loops which we compute instead, are expected to be the usual numbers, with smooth momentum dependence beyond perturbation theory.

Once again the large N matrix technology proved its power. Our matrix model is manifestly $O(d)$ invariant in any space dimension. Unlike the Eguchi-Kawai model[6], it has no matrix integrals, but rather algebraic equations. In a way, this is simpler, as we could use variational methods with some trial functions, dictated by physical intuition.

The Fock space of our theory is remarkably small. In fact, this is the very similar to we would get in the Eguchi-Kawai model. This was the model of $d = 4$ random matrices with some invariant probability measure. Treating this model by means of the operator expansion of the noncommutative probability theory we would get the similar Fock space. In four dimensions this a space of all sentences made from the four letter words – an infinite space, but still a discrete one.

There is a significant difference though. The EK model was the lattice model, with inevitable large N phase transition. The operator expansion in this model would correspond to the strong coupling expansion of original lattice gauge theory. Here, on the contrary, the local limit is already taken. We are dealing with continuum theory, and we are expanding the Lorentz invariant amplitudes in the low energy-momentum limit.

In terms of the Wilson loops, these amplitudes are given by path integrals over all paths, with Fourier measure (not the Wiener measure!). The absolute convergence of these integrals is to be achieved only at the nonperturbative level, due to the area law. At small momenta, the dominant paths would cover the area of the order of inverse string tension, which would lead to the momentum expansion coefficients proportional to inverse powers of the string tension.

These are not the directly observable quantities, because we do not include the quark degrees of freedom at the endpoints of our string. Those are included later in the meson Hamiltonian we construct. There are two terms, the free particle one, plus the contribution from the master field. This field is given by the gradient of the delta function of the $\hat{X}$ operator. The matrix elements of such master field between the trial states generated by the vertex operators are well defined and calculable.

In the glueball sector we derived the linear wave equation which is supposed to describe the mass spectrum. Again, the coefficients of this equation are directly related to the coefficients of the operator expansion. Truncating this equation would result in the finite linear equation in the glueball sector. In a way, this is even simpler than the meson wave equation.

We did not carry through these calculations, bat rather sketched the method. Working out the details and deriving these wave equations would be the most urgent next step.

An obvious question is how this agrees with the 't Hooft solution [1] in two dimensions. I do not know the full answer to this question, but our meson Hamiltonian in the light cone gauge $t = x_+$ could be compared to the 't Hooft's one. Clearly, the interaction term in the 't Hooft's Hamiltonian corresponds to the string momentum operator P_- which is related to our position field as discussed above.

So, we could interpret the 't Hooft solution in our terms. The real question is how to get this solution from our equations. This would give us an insight for the solution of the 4D problem. The momentum loop equation can be studied in the light cone gauge, where all the dynamics reduces to Coulomb "gluons" with propagators $\frac{1}{p_-^2}$.

The most appropriate would be the mixed x_+, p_- representation, where the momentum loop equation must reproduce the Bethe-Salpeter equation for the ladder diagrams. As was pointed out by 't Hooft, the x_+, p_- propagation goes along the strip with fixed width in p_- direction. It would be an interesting exercise to push forward this analogy.

Regardless the 2D tests, the suggested approach could be tested in 4D by recovering the planar QCD graphs. The formal perturbation expansion follows from direct iterations of the momentum loop equation[12], assuming the bare vacuum as a zeroth approximation.

Moreover, the perturbative QCD with running coupling must be recoverable in our representation at large p , or small $\hat{X}$. This must be an interesting computation, making use of some WKB asymptotics.

Finally, why not try to numerically compute the hadron spectrum, truncating the operator expansion and using variational approach like the one in[15]? This is a hard problem, but with the algebraic formulation we are getting closer to its solution.

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A Functional Fourier Transform

Let us parametrize the loops by the angle θ which varies from 0 to 2π. The functional measure for the path is defined according to the scalar product

$$(A, B) = \oint \frac{d\theta}{2\pi} A(\theta) B(\theta) \tag{140}$$

which diagonalizes in the discrete Fourier representation (not to be mixed with functional Fourier transform!)

$$A(\theta) = \sum_{-\infty}^{+\infty} A_n e^{\imath n\theta};\ A_{-n} = A_n^{\star} \tag{141}$$

$$(A, B) = \sum_{-\infty}^{+\infty} A_n B_{-n} = A_0 B_0 + \sum_{1}^{\infty} a'_n b'_n + a''_n b''_n;\ a'_n = \sqrt{2}\Re A_n, a''_n = \sqrt{2}\Im A_n \tag{142}$$

The corresponding measure is given by an infinite product of the Euclidean measures for the imaginary and real parts of each Fourier component

$$DQ = d^d Q_0 \prod_{1}^{\infty} d^d q'_n d^d q''_n \tag{143}$$

The orthogonality of Fourier transformation could now be explicitly checked, as

$$\begin{aligned} &\int DC \exp\left(\imath \int d\theta C_\alpha(\theta)\left(A_\alpha(\theta) - B_\alpha(\theta)\right)\right) \qquad (144)\\ = &\int d^d C_0 \prod_{1}^{\infty} d^d c'_n d^d c''_n \exp\left(2\pi\imath\left(C_0\left(A_0 - B_0\right) + \sum_{1}^{\infty} c'_n\left(a'_n - b'_n\right) + c''_n\left(a''_n - b''_n\right)\right)\right)\\ = &\ \delta^d\left(A_0 - B_0\right)\prod_{1}^{\infty}\delta^d\left(a'_n - b'_n\right)\delta^d\left(a''_n - b''_n\right) \end{aligned}$$

Let us now check the parametric invariance

$$\theta \to f(\theta);\ f(2\pi) - f(0) = 2\pi;\ f'(\theta) > 0 \tag{145}$$

The functions $x(\theta)$ and $p(\theta)$ have zero dimension in a sense, that only their argument transforms

$$x(\theta) \to x\left(f(\theta)\right);\ p(\theta) \to p\left(f(\theta)\right) \tag{146}$$

The functions $x'(\theta)$ and $p'(\theta)$ have dimension 1

$$p'(\theta) \to f'(\theta)p'\left(f(\theta)\right);\ x'(\theta) \to f'(\theta)x'\left(f(\theta)\right) \tag{147}$$

The invariance of the measure is easy to check for infinitesimal reparametrization

$$f(\theta) = \theta + \epsilon(\theta);\ \epsilon(2\pi) = \epsilon(0) \tag{148}$$

which changes x and (x, x) as follows

$$\delta x(\theta) = \epsilon(\theta)x'(\theta);\ \delta(x, x) = \oint \frac{d\theta}{2\pi}\epsilon(\theta)2x_\alpha(\theta)x'_\alpha(\theta) = -\oint \frac{d\theta}{2\pi}\epsilon'(\theta)x_\alpha^2(\theta) \tag{149}$$

The corresponding Jacobian reduces to

$$1 - \oint d\theta\epsilon'(\theta) = 1 \tag{150}$$

in virtue of periodicity.

The computation of the functional determinants can be performed in terms of discrete Fourier harmonics

$$J = \int Dx\delta^d(x(0)) \exp\left(\frac{1}{2}\imath\, F_{\mu\nu} \int d\theta x_\mu(\theta)x'_\nu(\theta)\right) = \tag{151}$$
$$\prod_{n=1}^{\infty} \int d^d x'_n d^d x''_n \exp\left(2\pi\imath\; n\, F_{\mu\nu}\, (x'_n)_\mu\, (x''_n)_\nu\right).$$

This yields the infinite product

$$J = \prod_{n=1}^{\infty} \left(n^d\; |\det F|\right)^{-1}, \tag{152}$$

which we define by means of the ζ regularization

$$J = \lim_{\alpha\to 0} \exp\left(-\sum_{n=1}^{\infty} \ln\left(n^d\; |\det F|\right)\, \left(n^d\; |\det F|\right)^{-\alpha}\right) \tag{153}$$

This yields

$$J = const\; |\det F|^{-\zeta(0)} = const\sqrt{|\det F|} \tag{154}$$

In odd dimensions this is zero, while in even dimensions it gives the pfaffian of F.

The easiest way to obtain this result would be to note, that it comes from elimination of the zero mode. In the original form we could have locally rotated variables

$$x_\mu(\theta) \Rightarrow \Omega_{\mu\nu}x_\nu(\theta), \tag{155}$$

to reduce the antisymmetric matrix to a canonical Jordan form

$$\frac{1}{2}\oint F_{\mu\nu}x_\mu dx_\nu \Rightarrow \sum_{k=1}^{\frac{1}{2}d} f_k \oint p_k dq_k \tag{156}$$

Then we could rescale the coordinates

$$p_k \Rightarrow \frac{p_k}{\sqrt{f_k}}, q_k \Rightarrow \frac{q_k}{\sqrt{f_k}} \tag{157}$$

after which the matrix F would disappear from the exponential. Only the delta function would acquire the corresponding Jacobian

$$\delta(x(0)) \Rightarrow \prod_{k=1}^{\frac{1}{2}d} f_k\delta(x(0)) = \sqrt{\det F}\delta(x(0)). \tag{158}$$

This is the correct factor, so the remaining jacobian from rescaling of the variables in the measure should cancel.

This is, indeed, so. Formally, the jacobian of this transformation is

$$\left(\det\sqrt{F}\right)^{-\delta(0)}. \tag{159}$$

where $\delta(\theta)$ is the periodic delta function

$$\delta(\theta) = \sum_{-\infty}^{\infty} \exp(\imath\, n\theta) = 1 + 2\sum_{1}^{\infty} \cos(n\theta). \tag{160}$$

Its limit at $\theta \to 0$ can be defined by analytic regularization

$$\delta_\alpha(\theta) = 1 + 2\sum_{1}^{\infty} n^{-\alpha}\,\cos(n\theta). \tag{161}$$

We find

$$\delta_\alpha(0) = 1 + 2\zeta(\alpha) \tag{162}$$

which in the limit $\alpha \to 0$ yields zero, as it should. This result means, that our measure with analytic regularization is scale invariant $x(\theta) \Rightarrow \lambda x(\theta)$ in addition to parametric invariance $x(\theta) \Rightarrow x(f(\theta))$.

B Numerical Approach

For the numerical solution these relations could be globally fitted by some variational Anzatz for $\hat{X}_\mu$ with finite large N, like it was done before[15] for the discrete version of the momentum loop equation.

The discretization used in that work also corresponded to a polygon in momentum space, but the discrete equations were valid only up to the powerlike corrections at large number L of vertexes of this polygon. The influence of this discretization on the renormalizability of QCD is yet to be studied theoretically. It could be that any approximation of this kind to the singular integral equations destroys asymptotic freedom.

The trial function was the Gaussian,

$$\begin{aligned} M(k_1, \dots k_L) &= \int Dx\, \exp\left(-\frac{1}{4}\int_0^T \dot{x}^2\right) W(x)\, \exp\left(-\imath \sum k_l x(t_l)\right) \\ &\approx Z_L \exp\left(-\sum_{\omega=2\pi n/T} |p_\omega|^2 Q(\omega)\right). \end{aligned} \tag{163}$$

The discrete Fourier expansion at the loop was used.

$$p(t) = \sum_{l=1}^{L} k_l \theta(t - t_l) = \frac{1}{T}\sum_{\omega=2\pi n/T} e^{\imath\,\omega t} p_\omega,\ p_{-\omega} = p^\star_\omega, p_0 = 0. \tag{164}$$

The variational function $Q(\omega)$ was parametrized by eight parameters (see[15]).

The Regge slope in this model is given by

$$\alpha' = \frac{Q'(0)}{2\pi}. \tag{165}$$

The discrepancy of the momentum loop equation was squared, analytically integrated over all momenta (with the Gaussian cutoff $\exp\left(-\sum_{t=1}^{L}\frac{p^2(t)}{\Lambda^2}\right)$) and then minimized for $L = 3, 4, ...$ The numerical limit $L = \infty$ was reached at $L \sim 40$.

Numerically, the renormalization group behaviour

$$\Lambda^2\alpha' \propto \exp\left(\frac{48\pi^2}{11Ng^2} + \frac{102}{121}\log Ng^2\right), \tag{166}$$

was shown to agree with the variational solution of the discretized momentum loop equation, but the accuracy in equation (about 0.1%) was still not sufficient to *derive* renormalization group behaviour numerically (see[15] for more details).

The important advantage of the present set of equations for the dual amplitudes is that *no approximations were made*. These are exact equations, therefore, the variational solution with the general enough Anzatz must reproduce asymptotic freedom. For example, one may try the above WKB approximation with adjustable running coupling.

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